Inverse Laplace Transform

reference: Fundamentals of Complex Analysis with Applications to Engineering and Science, by Saff and Snider

first let us consider <u>Fourier series</u>

Let $f(t)$ be L-periodic; that is,

$f(t+L) = f(t)$ for all t. Then it can be represented as a sum of complex exponentials (sin's and cos's) with angular frequency

$$\omega_n = \frac{2\pi n}{L}, \quad n = 0, \pm 1, \pm 2, \ldots$$

That is,

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n t} \qquad (1)$$

note:

$$f(t+L) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n(t+L)}$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n t} e^{i\omega_n L}$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n t} \underbrace{e^{i2\pi n}}_{=1}$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n t} = f(t).$$

how to find c_n?

multiply both sides of (1) by $e^{-i\omega_m t}$.

$$f(t) e^{-i\omega_m t} = \sum_{n=-\infty}^{\infty} c_n e^{+i(\omega_n-\omega_m)t}$$

$$\int_{-L/2}^{L/2} f(t) e^{-i\omega_m t}\, dt = \sum_{n=-\infty}^{\infty} c_n \int_{-L/2}^{L/2} e^{i(\omega_n-\omega_m)t}\, dt. \qquad (2)$$

now

$$\int_{-L/2}^{L/2} e^{i(\omega_n-\omega_m)t}\,dt = \begin{cases} \left.\dfrac{e^{i(\omega_n-\omega_m)t}}{i(\omega_n-\omega_m)}\right|_{-L/2}^{L/2} & n \neq m \\ L & n = m \end{cases}$$

now

$$e^{i(\omega_n-\omega_m)\frac{L}{2}} = e^{i(n-m)\frac{2\pi}{L}\cdot\frac{L}{2}} = e^{i(n-m)\pi}$$

$$e^{-i(\omega_n-\omega_m)\frac{L}{2}} = e^{i(m-n)\frac{2\pi}{L}\cdot\frac{L}{2}} = e^{i(m-n)\pi}$$

$$e^{i(n-m)\pi} - e^{-i(n-m)\pi} = e^{i(n-m)\pi}\left[1 - e^{-2i(n-m)\pi}\right]$$

since

$$e^{i2k\pi} = 1 \quad \text{when } k \text{ is an integer,}$$

$$e^{i(n-m)\pi} - e^{-i(n-m)\pi} = 0$$

$$\Rightarrow \int_{-L/2}^{L/2} e^{i(\omega_n - \omega_m)t}\, dt = \begin{cases} 0 & m \neq n \\ L & m = n. \end{cases}$$

back to (2):

$$\int_{-L/2}^{L/2} f(t) e^{-i\omega_m t}\, dt = c_m L$$

$$\Rightarrow \boxed{c_m = \frac{1}{L} \int_{-L/2}^{L/2} f(t) e^{-i\frac{2\pi m}{L} t}\, dt} \quad (3)$$

So Fourier series representation of $f(t)$, with $f(t) = f(t+L)\ \forall t$, is

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi n}{L} t},$$

with c_n given in (3).

now consider $L \to \infty$, in which case $f(t)$ approaches a non-periodic function defined on $(-\infty, \infty)$.

with (3), we write (1) as

$$f(t) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{L} \int_{-L/2}^{L/2} f(t)\, e^{-i\frac{2\pi n}{L}t}\, dt \right] e^{i\frac{2\pi n}{L}t}$$

since

$$\omega_{n+1} - \omega_n = \frac{2\pi(n+1)}{L} - \frac{2\pi n}{L} = \frac{2\pi}{L},$$

$$f(t) = \sum_{n=-\infty}^{\infty} \left[\frac{\omega_{n+1} - \omega_n}{2\pi} \int_{-L/2}^{L/2} f(t)\, e^{-i\frac{2\pi n}{L}t}\, dt \right] e^{i\frac{2\pi n}{L}t}.$$

define

$$g_L(\omega) = \frac{1}{2\pi} \int_{-L/2}^{L/2} f(t)\, e^{-i\omega t}\, dt.$$

so that

$$f(t) = \sum_{n=-\infty}^{\infty} g_L(\omega_n) e^{i\omega_n t} (\omega_{n+1} - \omega_n) \qquad (4)$$

now as $L \to \infty$,

$$g_L(\omega) \to \boxed{g(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt} \qquad (5)$$

$g(\omega)$ is <u>Fourier transform</u> of $f(t)$.

now as $L \to \infty$,

$$\omega_{n+1} - \omega_n = \frac{2\pi}{L} \to 0$$

then (4) is a Riemann sum of

$$\boxed{f(t) = \int_{-\infty}^{\infty} g(\omega) e^{i\omega t} d\omega} \qquad (6).$$

$f(t)$ is the <u>inverse Fourier transform</u>

now Laplace: suppose we have

$f(t) = 0$ for all $t < 0$; then from (5)

$$g(w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\, e^{-iwt}\, dt$$

$$= \frac{1}{2\pi}\int_{0}^{\infty} f(t)\, e^{-iwt}\, dt.$$

$g(w)$ is analytic in the lower half ~~plane~~ / the ~~imaginery~~ complex plane, since

$\left|e^{-iwt}\right| < 1$ when $\mathrm{Im}(w) < 0$. So let

$w = -is$, <u>s real and non-negative.</u>

~~we~~ We then have

$$g(-is) = \frac{1}{2\pi}\int_{0}^{\infty} f(t)\, e^{-st}\, dt.$$

define

$$F(s) = 2\pi g(-is):$$

$$\boxed{F(s) = \int_0^\infty f(t)\, e^{-st}\, dt} \qquad (7)$$

↑ Laplace transform of $f(t)$.

now inverse:

we have from (6)

$$f(t) = \int_{-\infty}^{\infty} g(\omega) e^{i\omega t}\, d\omega$$

and

$$g(\omega) = \frac{1}{2\pi} \underbrace{\boxed{\int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt.}}_{F(i\omega)}$$

↑ $F(i\omega)$
(replace s with $i\omega$ in (7)).

then with $g(\omega) = \frac{1}{2\pi} F(i\omega)$ in (6)

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(i\omega)\, e^{+i\omega t}\, d\omega$$

replace $i\omega \to s$ then

$$d\omega = \frac{1}{i} ds$$

$$s : -i\infty \to i\infty$$

$$f(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} F(s)\, e^{st}\, ds$$

(" ∞ is taken to be real ").

recall we assumed $s \geq 0$. ~~so~~ what if we have function ~~that we had a fu~~ for which we require $s > a$? That is, we require $a > 0$ sufficiently large so that

$$f(t)\, e^{-at}$$

is integrable.

then

$$f(t)e^{-at} = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \underbrace{\mathcal{L}(f(t)e^{-at})}_{\substack{= F(s+a) \\ \text{by translation thm.}}} e^{st}\,ds$$

$$= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} F(s+a)e^{st}\,ds$$

so

$$f(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} F(s+a)\,e^{(s+a)t}\,ds$$

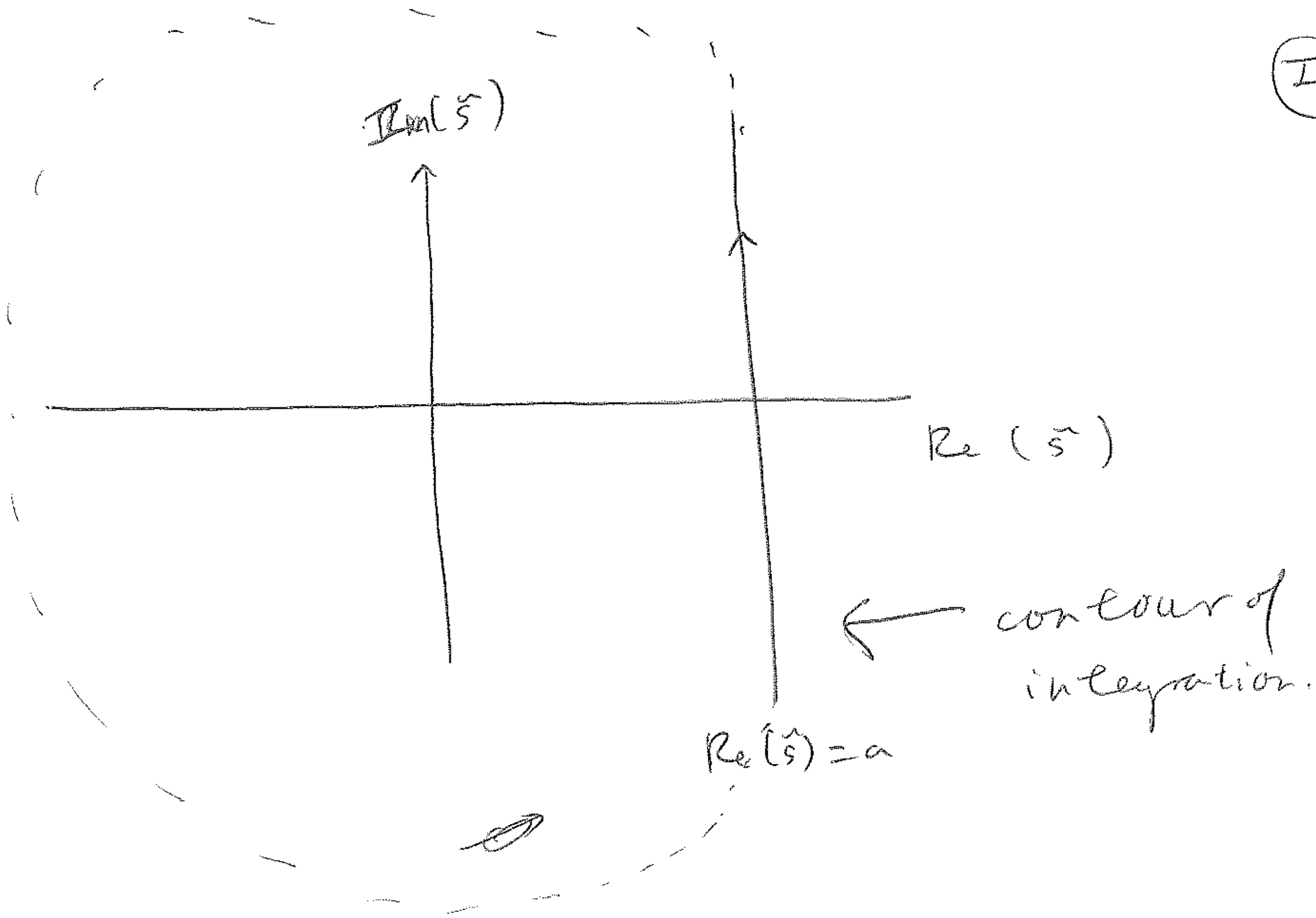
let

$$\tilde{s} = s+a, \text{ the}$$

$$\tilde{s} : a - i\infty \to a + i\infty.$$

$$\boxed{f(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} F(\tilde{s})e^{\tilde{s}t}\,d\tilde{s}}$$

Bromwich integral.



to calculate Bromwich integral, enclose the contour, apply Jordan's Lemma, and then use residue theorem to find

$$\boxed{f(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} F(\tilde{s})e^{\tilde{s}t}\,d\tilde{s} = \sum_{k} \mathrm{Res}(\tilde{s}_k)}$$

where

$\mathrm{Res}(\tilde{s}_k)$ are residues of

$F(\tilde{s})e^{\tilde{s}t}$