

The rescaling examples from class

EX

$$D \frac{d^2 w}{dx^2} + 1 = 0$$

$$w'(0) = 0$$

$$w(L) = 0$$

$$D = \lim_{\substack{\Delta x \rightarrow 0 \\ \Delta t \rightarrow 0}} \frac{\Delta x^2}{2 \Delta t}$$

$$[D] = \frac{L^2}{T}$$

$$[L] = L$$

$$[w] = T$$

rescaling reduces the number of effective ^{parameters} and also makes the ~~diff~~ equation dimensionless.

so we let $0 < x < L$ $0 < yL < L \Rightarrow 0 < y < 1$

$$x = yL \quad \begin{matrix} \nearrow \\ 0 < y < 1 \end{matrix} \quad (y = x/L)$$

notice that y has is dimensionless

now we let $w(x) = v(x/L) = v(y)$

then

$$\frac{dw}{dx} = \frac{dv}{dy} \frac{dy}{dx} = \frac{1}{L} \frac{dv}{dy}$$

$$\frac{d^2 w}{dx^2} = \frac{1}{L} \frac{d}{dy} \left(\frac{dv}{dy} \right) \frac{dy}{dx} = \frac{1}{L^2} \frac{d^2 v}{dy^2}$$

then we have

$$\frac{D}{L^2} \frac{d^2 v}{dy^2} + 1 = 0, \quad v'(0) = 0$$

$$v(1) = 0$$

notice: $[D/L^2] = \frac{L^2}{T} \frac{1}{L^2} = \frac{1}{T}$

$$[v] = T, \quad [y] = 1$$

lastly, let $u = \frac{D}{L^2} v \quad [u] = 1$

$$\Rightarrow \frac{D}{L^2} \cdot \frac{L^2}{D} \frac{d^2 u}{dy^2} + 1 = 0 \quad u'(0) = u(1) = 0$$

$$\Rightarrow u'' + 1 = 0, \quad u'(0) = 0, \quad u(1) = 0$$

this equation now has no parameters instead of 2, and is dimensionless

Ex

$$\frac{dy}{dt} = ry(k-y) \quad y(0) = y_0$$

$r, k > 0$ (3 parameters)

let $[y] = N$ (some unit)

$$\left. \begin{aligned} [t] &= T, & [k] &= N \\ [r] &= \frac{1}{NT}, & [y_0] &= N \end{aligned} \right\} \text{(must be the same as } [y] \text{)}$$

let $y = ku \Rightarrow [u] = 1$

then

$$\frac{dy}{dt} = k \frac{du}{dt} = rku(k - ku),$$

$$ku(0) = y_0$$

so we have

$$\frac{du}{dt} = rku(1-u), \quad u(0) = y_0/k$$

let $T = rkt$ [notice $[T] = \frac{1}{NT} \cdot N \cdot T = 1$]

and let $u(t) = w(T)$

then

$$\frac{du}{dt} = \frac{dw}{dT} \frac{dT}{dt} = rk \frac{dw}{dT}$$

so $rk \frac{dw}{dT} = rk w(1-w) \quad w(0) = y_0/k$

finally we have

$$\frac{dw}{dT} = w(1-w), \quad w(0) = \alpha$$

where $\alpha = y_0/k$ is the only parameter left in the problem.

This equation is now also ^{now} dimensionless