MATH 2120 – Midterm   October 30, 2014

- Duration: 80 minutes.
- This exam has 6 questions; **do 5 of the questions**. Each is worth 10 points.
- If you do all 6 questions, **indicate on the front of your exam booklet which 5 questions you would like to be graded**.
- Use of notes or calculators is not allowed.
- Show all your work and write legibly.
- Only work done in the exam booklet will be graded - if you run out of room, please ask for another and write your name on both booklets.

1. Consider the initial value problem

$$ty' + (t+1)\,y = 2\,t\,e^{-t}\,, \qquad y(1) = a\,. \tag{1}$$

Here, a is a real parameter that can take on any value.

(a) Write (1) in standard form.

(b) Compute the integrating factor. No derivation is required.

(c) Solve (1).

(d) There exists a value $a = a_0$ for which $a < a_0$ and $a > a_0$ produce qualitatively different behaviors as $t \to 0$. Determine the value of a_0.

(e) Determine the corresponding behavior of y as $t \to 0$ when $a < a_0$, $a = a_0$, and $a > a_0$.

2. Consider the first order differential equation

$$\frac{dy}{dx} = \frac{e^x}{2(y-1)}\,, \qquad y(1) = 0\,. \tag{2}$$

(a) Disregarding the initial condition, find an implicit general solution of (2).

(b) Find *the unique explicit* solution of (2) that satisfies the initial condition.

(c) For what range of x does the solution in (b) exist?

(d) Suppose the initial condition is taken instead to be $y(x_0) = y_0$. For what pair(s) (x_0, y_0) would the solution then be non-unique?

3. For the linear second order equation

$$y'' + p(t)\,y' + q(t)\,y = 0\,,$$

show that the Wronskian $W(y_1, y_2; t) = y_1 y_2' - y_2 y_1'$ of two solutions $y_1(t)$ and $y_2(t)$ is either always zero or never zero. Here, assume that $p(t)$ and $q(t)$ are continuous and differentiable for all t. **Show all steps of the derivation of the ODE for W. No points will be given for a memorized answer.**

4. Solve the initial value problem

$$y'' - 4y' + 5y = 0\,, \qquad y(4) = 0\,, \quad y'(4) = \beta\,,$$

where the primes denote differentiation with respect to the independent variable t. Here, β is a parameter that can take on all real values. Justify all steps.

5. Consider the initial value problem

$$y'' + 9y = \begin{cases} 9(t-1)\,, & 0 \le t \le 1\,, \\ \log t\,, & t > 1\,, \end{cases} \qquad y(0) = 1, \quad y'(0) = 0\,.$$

The solution for $y(t)$ may be written as

$$y(t) = \begin{cases} y_l(t)\,, & 0 \le t \le 1\,, \\ y_r(t)\,, & t > 1\,. \end{cases}$$

(a) Write the ODE **and** initial conditions (actual explicit values) for $y_r(t)$ so that the solution y is continuous and differentiable for all $t > 0$ (i.e., y and y' are continuous everywhere). State the method you would use to solve the resulting equation for $y_r(t)$, but **do not** solve it.

(b) Is y'' continuous? Explain.

6. Write the form of the particular solution for the following nonhomogeneous equations. All primes denote differentiation with respect to t, and $i = \sqrt{-1}$. **Do not solve for the coefficients.**

(a) $y'' - (1+i)\,y' + iy = t\sin t$

(b) $y'' - y = te^t \cosh t$