MATH 2120 – Midterm 1   October 15, 2013

- Duration: 80 minutes.
- This exam has 9 questions. Each is worth 10 points.
- Use of notes or calculators is not allowed.
- Show all your work and write legibly.
- Only work done in the exam booklet will be graded - if you run out of room, please ask for another and write your name on both booklets.
- Question (4a): find the stability of only **one** of the equilibrium points.
- Question (4b): not much work is required, but drawing a plot may help.
- Question (4c): while an explicit solution is easily obtainable, leave the solution in implicit form.
- Question (5b): **do not** solve the equation.
- Question (7): **do not** solve for the coefficients.
- May or may not be helpful: as $x \to 0$,

$$\sin x \sim x - \frac{x^3}{3!} \, ,$$

and

$$\cos x \sim 1 - \frac{x^2}{2!} \, .$$

1. Consider the equation

$$\frac{dy}{dt} + h(t)y = v(t)\,. \tag{1}$$

   (a) [7 pts] Derive the differential equation for the integrating factor and solve for it, thus obtaining the integrating factor. Please show all your work and include all steps - no credit will be given for a memorized answer.

   (b) [3 pts] Using the result in (a), find the general solution of (1) explicitly in terms of t. Please show all your work and include all steps - no credit will be given for a memorized answer.

2. Consider the initial value problem

$$t\frac{dy}{dt} + 2y = \frac{1}{t}\sin t\,, \qquad y(-\pi/2) = y_0\,, \quad t < 0\,. \tag{2}$$

   (a) [5 pts] For some arbitrary y_0, solve (2) for $y(t)$.

   (b) [5 pts] For what value of $y_0 = y_0^*$ is the behavior of $y(t)$ qualitatively different as $t \to 0^-$ versus when $y_0 \neq y_0^*$?

3. Consider the initial value problem

$$\frac{dy}{dt} = \frac{H_1'(t)}{H_2'(y)}\,, \qquad y(t_0) = y_0\,,$$

   where the primes denote differentiation with respect to the functions' own argument. Assume that both $H_1'(t)$ and $1/H_2'(y)$ and their derivatives are continuous for all t. Using the chain rule of differentiation (or any other method you deem justifiable), derive the result

$$H_2(y) = H_2(y_0) + H_1(t) - H_1(t_0)\,.$$

   Please show all your work and justify all steps - do not simply move dy's and dt's around.

4. Consider the equation

$$\frac{dy}{dt} = y^2 - 6y + 8\,. \tag{3}$$

   (a) [4 pts] Determine the stability of **one** of the equilibrium points by either writing $x = x_0 + \delta(t)$, $|\delta| \ll 1$, and solving the linearized equation for $\delta(t)$ (as in class), or using a result from homework.

   (b) [3 pts] For **each** initial condition $y(1) = 0$, $y(1) = 3$, $y(1) = 4$, state the behavior of y as $t \to \infty$. Not much work is required, but drawing a plot may help.

   (c) [3 pts] Find the general solution of (3). **Leave in implicit form.**

5. Consider the homogeneous first order equation

$$\frac{dy}{dx} = f(x, y) = F(y/x)\,. \tag{4}$$

(a) [7 pts] Making the substitution $y(x) = xv(x)$, separate variables to write the solution in the form

$$\int G(v)\,dv = \int H(x)\,dx\,,$$

for some functions G (in terms of F), and H. No credit will be given for a memorized answer.

(b) [3 pts] For

$$f(x, y) = \frac{x + y}{x - x^2/y}$$

in (4), write $f(x, y)$ in the form $f(x, y) = F(v)$, where $y(x) = xv(x)$. That is, find $F(v)$. **Do not** solve the equation.

6. Solve the initial value problem

$$y'' - 2y' + 5y = 0\,, \qquad y(3) = 0\,, \quad y'(3) = \beta\,,$$

where the primes denote differentiation with respect to the independent variable t.

7. Write the form of the particular solution of the following equations (primes denote differentiation with respect to t). **Do not** solve for the coefficients.

(a) [5 pts]

$$y'' + 4y = t^2 e^{it} \cos t$$

(b) [5 pts]

$$y''' - 4y' = (2 + t^2)(e^{2t} + t)$$

8. For the equation

$$\frac{d^2y}{dx^2} - 2b\frac{dy}{dx} + b^2 y = P_m(x)e^{bx}\,,$$

where b is real and nonzero, explain how to obtain the form of the particular solution of $y(x)$ by interpreting $y(x)$ as the homogeneous solution of a higher order equation. Here, $P_m(x)$ is a polynomial of order m, where m is a positive integer. Then give the form for the particular solution.

9. For all real and strictly positive values of ω, find the general solution of

$$\frac{d^2x}{dt^2} + \omega^2 x = \sin t\,.$$

Make sure to treat the special value(s) of ω separately.