Midterm 1 solutions

① $$\frac{dy}{dt} + h(t)y = v(t)$$

$$u(t)\frac{dy}{dt} + u(t)h(t)y = u(t)v(t)$$

want LHS to be full derivative

$$\frac{d}{dt}(uy) = u'y + uy'$$

then

$$\boxed{\frac{du}{dt} = u(t)h(t)}$$

$$\int \frac{du}{u} = \int h(t)\,dt$$

$$\Rightarrow \boxed{u = e^{\int h(t)\,dt}}$$

b) $$\frac{d}{dt}(uy) = u(t)\,v(t)$$

$$uy = \int uv\,dt + c$$

$$y = \frac{1}{u}\left[\int uv\,dt + c\right]$$

$$\boxed{y = e^{-\int h(t)\,dt}\left[\int e^{\int h(t)\,dt}\,v(t)\,dt + c\right]}$$

2) $$t\frac{dy}{dt} + 2y = \frac{1}{t}\sin t, \quad y(-\pi/2) = y_0, \quad t<0.$$

$$\frac{dy}{dt} + \frac{2}{t}y = \frac{1}{t^2}\sin t$$

I.F. $$u(t) = e^{\int \frac{2}{t}\,dt} = e^{2\log t} = t^2$$

$$\frac{d}{dt}(t^2 y) = \sin t$$

$$t^2 y = -\cos t + C$$

$$y = \frac{-\cos t + C}{t^2}$$

$$y(-\pi/2) = \frac{0 + C}{\pi^2/4} = y_0$$

$$C = \pi^2 y_0/4$$

$$\Rightarrow \boxed{y = \frac{-\cos t + \pi^2 y_0/4}{t^2}}$$

as $t \to 0^-$, $\cos t \sim 1 - t^2/2$

thus if $\pi^2 y_0 / 4 = 1$, then

y has finite limit as $t \to 0^-$.

otherwise $y \to \infty$ as $t \to 0^-$

$$\Rightarrow \boxed{y_0^* = \frac{4}{\pi^2}}$$

(3). $$\frac{dy}{dt} = \frac{H_1'(t)}{H_2'(y)}, \quad y(t_0) = y_0$$

$$H_2'(y)\frac{dy}{dt} = H_1'(t)$$

chain rule.

$$\frac{d}{dt} H_2(y(t)) = H_1'(t)$$

integrate w.r.t. time

$$H_2(y) = H_1(t) + C$$

$$H_2(y_0) = H_1(t_0) + C \qquad (\text{I.C.})$$

$$C = H_2(y_0) - H_1(t_0)$$

$$\Rightarrow \quad H_2(y) = H_1(t) + H_2(y_0) - H_1(t_0)$$

(4) $$\frac{dy}{dt} = y^2 - 6y + 8 = (y-4)(y-2)$$

~~As~~ look at $y = 2$.

$$y = 2 + \delta, \quad \frac{dy}{dt} = \frac{d\delta}{dt}$$

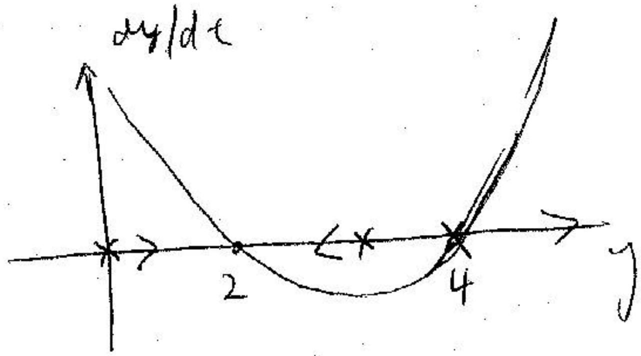
$$\frac{d\delta}{dt} = (2+\delta-2)(2+\delta-4)$$

$$= \delta(2+\delta-4) = -2\delta + O(\delta^2)$$

$$\Rightarrow \frac{d\delta}{dt} = -2\delta \Rightarrow \delta = ce^{-2t}$$

since $\delta \to 0$ as $t \to 0$, <u>stable</u>

b).



$y(1) = 0$: $y \to 2$.

$y(1) = 3$: $y \to 2$.

$y(1) = 4$: $y \to 4$ (actually $y \equiv 4$)

c) $$\frac{dy}{dt} = (y-2)(y-4)$$

$$\int \frac{dy}{(y-2)(y-4)} = \int dt$$

$$\frac{1}{(y-2)(y-4)} = \frac{A}{y-2} + \frac{B}{y-4}$$

$$= \frac{A(y-4) + B(y-2)}{(y-2)(y-4)}$$

$$= (A+B)y - 4A - 2B$$

$$A + B = 0, \quad -4A - 2B = 1$$

$$A = -B \Rightarrow -4A + 2A = 1$$

$$A = -1/2, \quad B = 1/2$$

$$\Rightarrow \frac{1}{2} \int \frac{1}{y-4} - \frac{1}{y-2} \, dy = t + C$$

$$\boxed{\frac{1}{2}\log\left(\frac{y-4}{y-2}\right) = t + C}$$

(implicit).

5 $\quad \dfrac{dy}{dx} = F(y/x) \quad$ let $y = xv$

then $\quad y' = v + xv'$

so

$$v + x\frac{dv}{dx} = F(v).$$

$$x\frac{dv}{dx} = F(v) - v.$$

$$\int \frac{dv}{F(v)-v} = \int \frac{dx}{x}$$

b) $f(x,y) = \dfrac{x+y}{x - x^2/y} = \dfrac{1 + y/x}{1 - x/y}$

then letting $y = xv$,

$$\boxed{F(v) = \frac{1+v}{1 - 1/v}}$$

6. $y'' - 2y' + 5y = 0$, $y(3) = 0$, $y'(3) = \beta$

$r^2 - 2r + 5 = 0$

$r = \dfrac{2 \pm \sqrt{4 - 20}}{2} = 1 \pm 2i$

$y = e^{t-3}\left[A\cos(2(t-3)) + B\sin(2(t-3))\right]$

$y(3) = 0 \Rightarrow A = 0.$

$y'(3) = \beta = 2B$ (since $\sin 0 = 0$)

$\Rightarrow B = \beta/2.$

$$\Rightarrow \boxed{y = e^{t-3} \cdot \frac{\beta}{2} \sin(2(t-3))}$$

(7)

a) $y'' + 4y = t^2 e^{it} \cos t$

$$= t^2 e^{it} \left[\frac{e^{it} + e^{-it}}{2} \right]$$

$$= \frac{t^2}{2} \left[e^{2it} + 1 \right]$$

$r_f = 2i$ $\quad$ $r_f = 0.$

$$r^2 + 4 = 0 \Rightarrow r = \pm 2i$$

$$\Rightarrow \boxed{y_p = \underbrace{a_0 + a_1 t + a_2 t^2}_{r_f = 0} + \underbrace{t\left(b_0 + b_1 t + b_2 t^2\right)e^{i2t}}_{r_f = 2i}}$$

b). $y''' - 4y' = (2+t^2)(e^{2t} + t)$

$$= \underbrace{(2+t^2)e^{2t}}_{r_f = 2} + \underbrace{t^3 + 2t}_{r_f = 0}$$

$r^3 - 4r = 0 \quad (r^2 - 4)r = 0$

$r = \pm 2, \; r = 0$

$$\Rightarrow \boxed{\begin{aligned} y_p &= t\left(a_0 + a_1 t + a_2 t^2\right)e^{2t} \quad \leftarrow r_f = 2 \\ &\quad + t\left[b_0 + b_1 t + b_2 t^2 + b_3 t^3\right] \quad \leftarrow r_f = 0 \end{aligned}}$$

(8) $$y'' - 2by' + b^2 y = P_m(x) e^{bx}$$

$$r^2 - 2br + b^2 = 0$$

$$r = \frac{2b \pm \sqrt{4b^2 - 4b^2}}{2} = b, b$$

$$\Rightarrow \left(\frac{d}{dx} - b\right)^2 y = P_m(x) e^{bx}$$

$$\left(\frac{d}{dx} - b\right)^{m+1} \left(\frac{d}{dx} - b\right)^2 y = 0.$$

$$\Rightarrow y = \Big(\underbrace{a_0 + a_1 x}_{\text{homog.}} \underbrace{\boxed{+ a_2 x^2 + \dots + a_{m+2} x^{m+2}}}_{y_p}\Big) e^{bx}$$

(9). $$\frac{d^2 x}{dt^2} + w^2 x = \sin t.$$

[illegible] $\quad r^2 + w^2 = 0 \Rightarrow r = \pm iw$

[illegible] $y_h = A\cos wt + B\sin wt$

[illegible] $w \neq 1$

$$y_p = a\cos t + b\sin t.$$

$$y_p' = -a\sin t + b\cos t.$$

$$y_p'' = -a\cos t - b\sin t.$$

$$-a\cos t - b\sin t + w^2(a\cos t + b\sin t) = \sin t$$

$$\Rightarrow -a + aw^2 = 0 \Rightarrow a = 0$$

$$-b + w^2 b = 1 \Rightarrow \boxed{b = \frac{1}{w^2-1}}$$

$$\Rightarrow y_p = \frac{1}{w^2-1}\sin t$$

then

$$y = y_h + y_p.$$

<u>$\omega = 1.$</u>

$$y_p = t\,(a\cos t + b\sin t).$$

$$y_p' = a\cos t + b\sin t + t\,(-a\sin t + b\cos t)$$

$$y_p'' = -a\sin t + b\cos t - a\sin t + b\cos t$$
$$+\, t\,[-a\cos t - b\sin t]$$

$$y_p'' + y_p = \sin t.$$

then

$$-2a\sin t + 2b\cos t = \sin t.$$

$$\Rightarrow b = 0, \quad a = \frac{-1}{2} \qquad \Big| \Rightarrow y = y_h + y_p.$$
$$\Rightarrow y_p = \frac{-1}{2}\, t\cos t$$