MATH 2120 – Midterm 2 November 14, 2013

- Duration: 80 minutes.
- A table of Laplace transforms is included as the final page.
- This exam has 4 questions. Each is worth 10 points.
- Question 5 is a bonus question worth 6 points. Part marks will be given.
- Use of notes or calculators is not allowed.
- Show all your work and write legibly.
- No credit will be given for memorized answers.
- Only work done in the exam booklet will be graded - if you run out of room, please ask for another and write your name on both booklets.
- Laplace transform of $f(t)$:
$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt\,.$$
- Laplace transform of $f'(t)$:
$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)\,.$$
- Laplace transform of $f''(t)$:
$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)\,.$$

1. If $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of the homogeneous equation $y''+p(x)y'+q(x)y=0$, then a particular solution $y_p(x)$ of the nonhomogeneous equation

$$y'' + p(x)y' + q(x)y = g(x)\,,$$

is

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x)\,, \tag{1}$$

where

$$c_1(x) = -\int \frac{y_2(x)g(x)}{W(x;y_1,y_2)}\,dx\,, \qquad c_2(x) = \int \frac{y_1(x)g(x)}{W(x;y_1,y_2)}\,dx\,; \qquad W(x;y_1,y_2) \equiv y_1y_2' - y_2y_1'\,. \tag{2}$$

Use (1) and (2) to find a particular solution of equation

$$xy'' + (5x-1)y' - 5y = x^2e^{-5x}\,, \qquad x > 0\,, \tag{3}$$

given that $y_1 = 5x-1$ and $y_2 = e^{-5x}$ are two linearly independent solutions of the corresponding homogeneous equation. Identify any homogeneous terms (if any) that result from your calculations.

2. (a) Show that

$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)\,,$$

where $F(s) = \mathcal{L}\{f(t)\}$.

(b) Calculate the Laplace transform

$$\mathcal{L}\{\cos^2(at)\} = \int_0^\infty e^{-st}\cos^2(at)\,dt\,, \tag{4}$$

by direct computation of the integral (or equivalent integrals). Do not use the Laplace transform table or any theorems and results from class. **What restriction must be placed on s in order for the integral in** (4) **to converge? Explain.**

3. Use the method of Laplace transforms to solve the initial value problem

$$y'' + 6y' + 13 = e^{3t}\,, \qquad y(0) = 0, \qquad y'(0) = 0\,.$$

If you solve it without using Laplace transforms, you will receive a maximum of $5/10$.

4. Use the method of Laplace transforms to solve the initial value problem

$$y'' + 4y = t, \qquad y(0) = \alpha, \qquad y'(0) = \beta\,.$$

You will not receive credit for solving it without using Laplace transforms.

Optional Bonus Question (6 points)

5. (a) [2 pts] Derive $c_1(x)$ in (2).

   (b) Consider the equation for a forced damped oscillator with mass $m = 2$ and spring constant $k = 3$,

$$2x'' + cx' + 3x = \cos\omega t\,, \tag{5}$$

   with some initial conditions.

   i. [1 pt] For $c > 0$, explain why, for large time, only the particular solution of (5) is relevant.

   ii. [1 pt] The oscillation amplitude $C(\omega)$ of the particular solution is

$$C(\omega) = \frac{1}{\sqrt{(3-2\omega^2)^2 + (c\omega)^2}}\,.$$

   For what range of $c > 0$ is there a practical resonance frequency $\omega = \omega_{max}$? For that range of c, find ω_{max}.

   iii. [1 pt] Now let $c = 0$ and $x(0) = x'(0) = 0$ in (5). The full solution of (5) is then

$$x(t) = \frac{1}{2(\omega_0^2 - \omega^2)}\left(\cos\omega t - \cos\omega_0 t\right)\,, \tag{6}$$

   where $\omega_0 = \sqrt{3/2}$ is the natural frequency. For ω close to ω_0, use the identity

$$2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$$

   to write (6) in a form that indicates rapid oscillations within a slowly varying envelope. **Identify which term corresponds to the slowly varying envelope, and which corresponds to rapid oscillations.**

   iv. [1 pt] For $c = 0$, at what value of ω does $C(\omega)$ become unbounded? That is, what is the resonant frequency of (5) when $c = 0$? Can $C(\omega)$ ever become unbounded when $c > 0$? **Explain**.

Table of Laplace Transforms

This table summarizes the general properties of Laplace transforms and the Laplace transforms of particular functions derived in Chapter 4.

Function	Transform	Function	Transform
$f(t)$	$F(s)$	e^{at}	$\dfrac{1}{s-a}$
$af(t)+bg(t)$	$aF(s)+bG(s)$	$t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$
$f'(t)$	$sF(s)-f(0)$	$\cos kt$	$\dfrac{s}{s^2+k^2}$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$	$\sin kt$	$\dfrac{k}{s^2+k^2}$
$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$	$\cosh kt$	$\dfrac{s}{s^2-k^2}$
$\int_0^t f(\tau)\,d\tau$	$\dfrac{F(s)}{s}$	$\sinh kt$	$\dfrac{k}{s^2-k^2}$
$e^{at}f(t)$	$F(s-a)$	$e^{at}\cos kt$	$\dfrac{s-a}{(s-a)^2+k^2}$
$u(t-a)f(t-a)$	$e^{-as}F(s)$	$e^{at}\sin kt$	$\dfrac{k}{(s-a)^2+k^2}$
$\int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)G(s)$	$\dfrac{1}{2k^3}(\sin kt-kt\cos kt)$	$\dfrac{1}{(s^2+k^2)^2}$
$tf(t)$	$-F'(s)$	$\dfrac{t}{2k}\sin kt$	$\dfrac{s}{(s^2+k^2)^2}$
$t^nf(t)$	$(-1)^nF^{(n)}(s)$	$\dfrac{1}{2k}(\sin kt+kt\cos kt)$	$\dfrac{s^2}{(s^2+k^2)^2}$
$\dfrac{f(t)}{t}$	$\int_s^\infty F(\sigma)\,d\sigma$	$u(t-a)$	$\dfrac{e^{-as}}{s}$
$f(t)$, period p	$\dfrac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)\,dt$	$\delta(t-a)$	e^{-as}
1	$\dfrac{1}{s}$	$(-1)^{[\![t/a]\!]}$ (square wave)	$\dfrac{1}{s}\tanh\dfrac{as}{2}$
t	$\dfrac{1}{s^2}$	$\left[\!\left[\dfrac{t}{a}\right]\!\right]$ (staircase)	$\dfrac{e^{-as}}{s(1-e^{-as})}$
t^n	$\dfrac{n!}{s^{n+1}}$		
$\dfrac{1}{\sqrt{\pi t}}$	$\dfrac{1}{\sqrt{s}}$		
t^a	$\dfrac{\Gamma(a+1)}{s^{a+1}}$		