Midterm 2 solutions

1) $xy'' + (5x-1)y' - 5y = x^2 e^{-5x}$

coeff must be 1 to apply VOP formula given

$$y'' + \frac{5x-1}{x}y' - \frac{5}{x}y = xe^{-5x} = g(x)$$

$$y_1 = 5x-1, \quad y_2 = e^{-5x}$$

$$W(x;y_1,y_2) = \begin{vmatrix} 5x-1 & e^{-5x} \\ 5 & -5e^{-5x} \end{vmatrix} = -5(5x-1)e^{-5x} - 5e^{-5x}$$

$$= -25xe^{-5x}$$

$$\frac{g(x)}{W(x;y_1,y_2)} = \frac{-1}{25}$$

$$c_1(x) = -\int y_2 \frac{g(x)}{W(x;y_1,y_2)}dx = \frac{+1}{25}\int e^{-5x}dx$$

$$= \frac{-1}{125}e^{-5x}$$

$$c_2(x) = +\int y_1 \frac{g(x)}{W(x;y_1,y_2)}dx = \frac{-1}{25}\int 5x-1\,dx$$

$$= \frac{-1}{25}\left(5\frac{x^2}{2} - x\right)$$

then

$$y_p = c_1 y_1 + c_2 y_2 = \frac{-1}{125} e^{-5x}(5x-1) + e^{-5x}\left(\frac{-1}{25}\left(5\frac{x^2}{2} - x\right)\right)$$

$$= \frac{-1}{25} x e^{-5x} + \frac{1}{125} e^{-5x} + e^{-5x}\left(\frac{x}{25} - \frac{x^2}{10}\right)$$

$$= \frac{1}{125} e^{-5x} - \frac{x^2}{10} e^{-5x}$$

homogeneous term.

$$\rightarrow \boxed{y_p = -\frac{x^2}{10} e^{-5x}}$$

(2) a)

$$\mathcal{L}(f''(t)) = \int_0^\infty e^{-st} \frac{d^2 f}{dt^2}\, dt$$

$$= e^{-st} \frac{df}{dt}\Big|_0^\infty + s\int_0^\infty e^{-st} \frac{df}{dt}\, dt$$

$$= e^{-st}\frac{df}{dt}\Big|_0^\infty + s\left[e^{-st}f(t)\Big|_0^\infty + s\underbrace{\int_0^\infty e^{-st} f\,dt}_{\mathcal{L}(f)}\right]$$

$$= -f'(0) + s\left[-f(0) + s^2 F(s)\right]$$

$$= s^2F(s) - sf(0) - f'(0)$$

(assuming f is of exponential order).

b) $$\mathcal{L}(\cos^2 at) = \int_0 e^{-st}\cos^2 at \, dt$$

$$= \int_0^\infty e^{-st}\left[\frac{1}{2}\left(e^{iat} + e^{-iat}\right)\right]^2 dt$$

$$= \frac{i}{4}\int_0^\infty e^{-st}\left[e^{i2at} + e^{-i2at} + 2\right] dt$$

$$= \frac{1}{4}\left[-\frac{e^{-(s-i2a)t}}{s-i2a} - \frac{e^{-(s+i2a)t}}{s+2ia} - \frac{2}{s}e^{-st}\right]_0^\infty$$

[require $\boxed{s > 0}$ for all exp. terms to decay at ∞]

$$= \frac{1}{4}\left[\frac{1}{s - i2a} + \frac{1}{s + i2a} + \frac{2}{s}\right]$$

$$= \frac{1}{4}\left[\frac{s + i2a + s - i2a}{s^2 + 4a^2} + \frac{2}{s}\right]$$

$$= \frac{1}{4}\left[\frac{2s \cdot s + 2(s^2 + 4a^2)}{s(s^2 + 4a^2)}\right]$$

$$= \frac{1}{4}\left[\frac{4s^2 + 8a^2}{s(s^2 + 4a^2)}\right]$$

$$= \frac{s^2 + 2a^2}{s(s^2 + 4a^2)}$$

$$\Rightarrow \boxed{\mathcal{L}(\cos^2 at) = \frac{s^2 + 2a^2}{s(s^2 + 4a^2)} \; ; \; s > 0}$$

(3) $y'' + 6y' + 13y = e^{3t}$, $y(0) = y'(0) = 0$

apply $\mathcal{L}(\cdot)$:

$$(s^2 + 6s + 13)Y(s) = \frac{1}{s-3}$$ (because IC's are 0).

$$Y(s) = \frac{1}{(s-3)(s^2+6s+13)} = \frac{A}{s-3} + \frac{Bs+C}{(s+3)^2+4}$$

then

$$1 = A\left[(s+3)^2+4\right] + (Bs+C)(s-3)$$

<u>s=3</u>

$$1 = A\left[6^2+4\right] = 40A \Rightarrow \underline{A = 1/40}$$

<u>coeff of s^2</u>

$$0 = A + B \Rightarrow \underline{B = -1/40}$$

coeff of s^0:

$$1 = 13A - 3C$$

$$1 = \frac{13}{40} - 3C \Rightarrow \frac{27}{40} = -3C$$

$$C = -\frac{9}{40}$$

then

$$Y(s) = \frac{1/40}{s-3} + \frac{(-1/40)s - 9/40}{(s+3)^2+4}$$

$$= \frac{1}{40}\left[\frac{1}{s-3} - \frac{s+9}{(s+3)^2+4}\right]$$

$$= \frac{1}{40}\left[\frac{1}{s-3} - \frac{s+3+6}{(s+3)^2+4}\right]$$

$$= \frac{1}{40}\left[\frac{1}{s-3} - \frac{s+3}{(s+3)^2+4} - \frac{3\cdot 2}{(s+3)^2+4}\right]$$

then $y(t) = \mathcal{L}^{-1}(Y(s))$

$$\boxed{y(t) = \frac{1}{40}\left[e^{3t} - e^{-3t}\cos 2t - 3e^{-3t}\sin 2t\right]}$$

(4) $y'' + 4y = t$; $y(0) = \alpha$, $y'(0) = \beta$.

apply $\mathcal{L}(\cdot)$:

$$s^2 Y(s) - s\,y(0) - y'(0) + 4Y(s) = \frac{1}{s^2}$$

$$(s^2+4)Y(s) = \frac{1}{s^2} + \alpha s + \beta$$

$$Y(s) = \frac{1}{s^2(s^2+4)} + \frac{\alpha s + \beta}{s^2+4}$$

$$= \frac{1}{s^2(s^2+4)} + \alpha\frac{s}{s^2+4} + \frac{\beta}{2}\,\frac{2}{s^2+4}$$

↑

expand this term in partial fractions.

$$\frac{1}{s^2(s^2+4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+4}$$

$$1 = (As+B)(s^2+4) + (Cs+D)s^2$$

$s=0$

$$1 = 4B \Rightarrow \underline{B = 1/4}$$

$s=2i$

$$1 = (2iC+D)(-4)$$

$$1 = -8iC - 4D \Rightarrow \underline{D = -1/4}$$

coeff of s

$$0 = A \Rightarrow \underline{A=0}$$

coeff of s^3

$$0 = A + C \Rightarrow \underline{C=0}$$

$$\Rightarrow Y(s) = \frac{1/4}{s^2} + \frac{(-1/4)}{s^2+4} + \alpha\frac{s}{s^2+4} + \frac{\beta}{2}\frac{2}{s^2+4}$$

$$y(t) = \frac{t}{4} - \frac{1}{8}\sin 2t + \alpha\cos 2t + \frac{\beta}{2}\sin 2t$$

$$\boxed{y(t) = \frac{t}{4} + \alpha\cos 2t + \left(\frac{\beta}{2} - \frac{1}{8}\right)\sin 2t}$$

BONUS

5 a) look at HW4 solutions (#2)

b)

i) $x(t) = x_h(t) + x_p(t)$

homogeneous particular

$X_h(t) = Ae^{rt}$;

$$2r^2 + cr + 3 = 0$$

$$r_{\pm} = \frac{-c \pm \sqrt{c^2 - 24}}{4}.$$

if real part of r is negative,

then

$x_h \to 0$ as $t \to \infty$ so that

$x(t) \to x_p(t)$ as $t \to \infty$.

if $c^2 < 24$ (under damped)

$\sqrt{c^2-24}$ is imaginary

then

$$\text{real}(r_\pm) = -\frac{c}{4} < 0.$$

if $c^2 > 24$ (over damped)

$$\text{real}(r_-) = \frac{-c - \sqrt{c^2-24}}{4} < 0$$

$$\text{real}(r_+) = \frac{-c + \sqrt{c^2-24}}{4}$$

since $c^2 - 24 < c^2$

$$\sqrt{c^2-24} < c$$

so

$$-c+\sqrt{c^2-24} < 0$$

$\Rightarrow$ real $(r_+) < 0$

So when $c > 0$, real part of $r < 0$

then $x_h \to 0$ as $t \to \infty$,

$\Rightarrow x \to x_p$ as $t \to \infty$.

ii) $$C(\omega) = \frac{1}{\sqrt{(3-2\omega^2)^2+(c\omega)^2}}$$

C is maximum when $\frac{dC}{d\omega} = 0$

~~dw~~

$$\frac{dC}{d\omega} = \frac{-1}{2(\ldots)^{3/2}}\left[2(3-2\omega^2)(-4\omega)+2c^2\omega\right] = 0$$

$$-4(3-2\omega^2)+c^2 = 0$$

$$8\omega^2 = 12 - c^2$$

$$\boxed{\omega_{max} = \left[\frac{1}{8}\left(12 - c^2\right)\right]^{1/2} \qquad \text{require } c^2 < 12 \text{ for } \omega_{max} \text{ to be real}}$$

iii) ~~$x(t)$~~

$$x(t) = \frac{1}{2(\omega_0^2 - \omega^2)}\left(\cos \omega t - \cos \omega_0 t\right)$$

since

$$2 \sin\alpha \sin\beta = \cos(\alpha - \beta) - \cos(\alpha + \beta)$$

identify

$$\alpha - \beta \to \omega t$$

$$\alpha + \beta \to \omega_0 t$$

so that

$$\alpha = \frac{1}{2}(\omega + \omega_0)t$$

$$\beta = \frac{1}{2}(\omega_0 - \omega)t.$$

then

$$x(t) = \boxed{\frac{1}{\omega_0^2-\omega^2}\sin\left(\tfrac{1}{2}(\omega_0-\omega)t\right)}\sin\left(\tfrac{1}{2}(\omega_0+\omega)t\right)$$

slowly varying envelope

rapid oscillations.

iv)

$$C(\omega) = \frac{1}{\sqrt{(3-2\omega^2)^2+(c\omega)^2}}.$$

when $c=0$, C unbounded when

$$3-2\omega^2 = 0$$

$$\boxed{\omega = \sqrt{3/2}.}$$

If $c>0$, C never unbounded since

$$(3-2\omega^2)^2 \geq 0, \quad (c\omega)^2 > 0$$

so

$$(3-2\omega^2)+(c\omega)^2 > 0 \text{ for all } \omega.$$