Impulses and Delta Functions (4.6)

- in terms of forces, the quantity

$$p = \int_a^b f(t)\,dt$$

where $f(t)$ is a force, is called the impulse and is equal to the change in momentum of the object on which it is acting.

- Newton Law :

$$m\frac{dv}{dt} = f(t)$$

$$\int_a^b m\frac{dv}{dt}\,dt = \int_a^b f(t)\,dt.$$

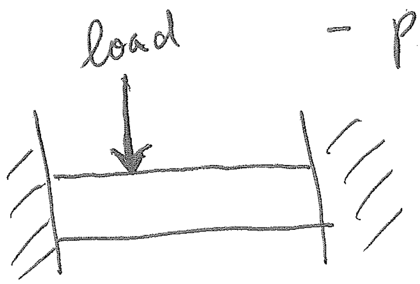
$$\Rightarrow \underbrace{mv(b) - mv(a)}_{\text{change in momentum over interval } [a,b]} = p.$$

- the Dirac delta function (δ- fcn) approximates an ~~[illegible]~~ very large force acting over a very short duration

- the impulse it generates is 1

<u>examples</u> - bat hitting a baseball

- point loading on a beam

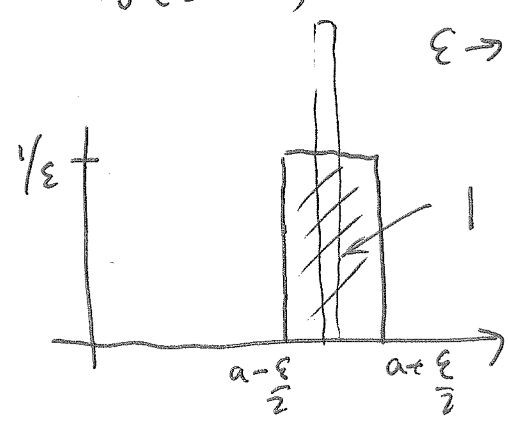


<u>Definitions of δ fcn</u>

- as a limit of a sequence of functions

$$\delta_a(t) = \delta(t-a) = \lim_{\varepsilon \to 0} \begin{cases} 1/\varepsilon & a-\frac{\varepsilon}{2} < t < a+\frac{\varepsilon}{2} \\ 0 & \text{otherwise} \end{cases}$$

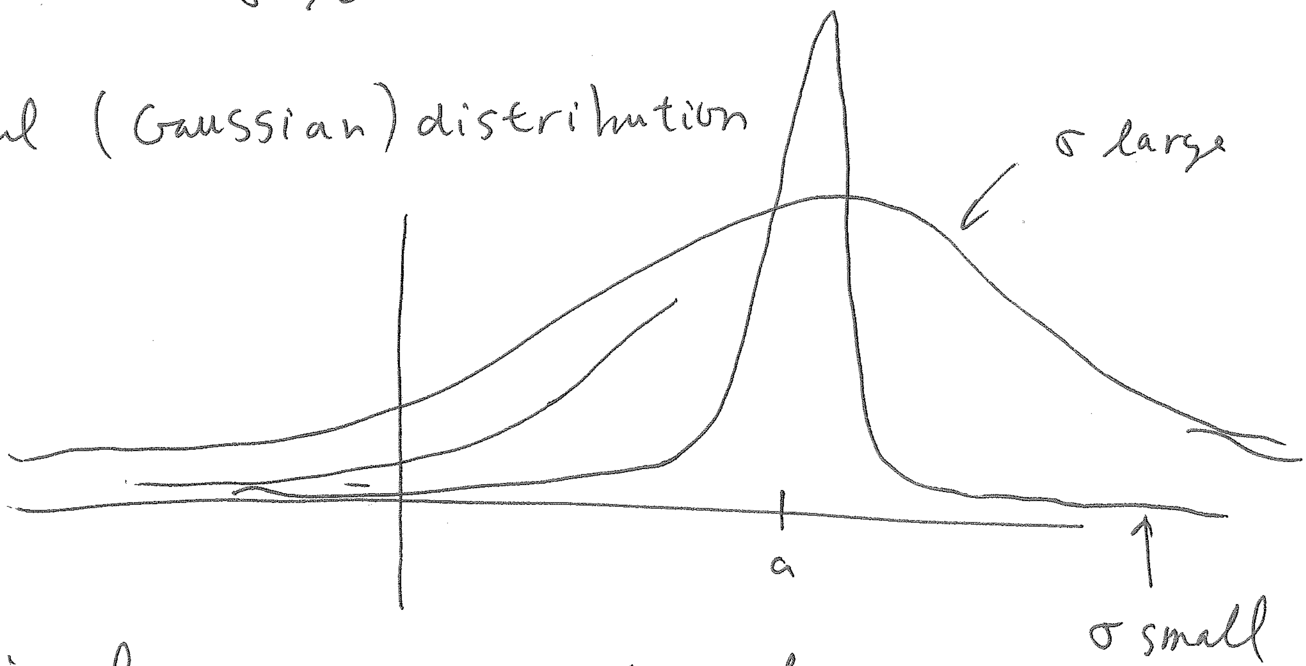
$\uparrow$ book



or

$$\delta(t-a) = \lim_{\sigma \to 0} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-a)^2}{2\sigma^2}}$$

Normal (Gaussian) distribution



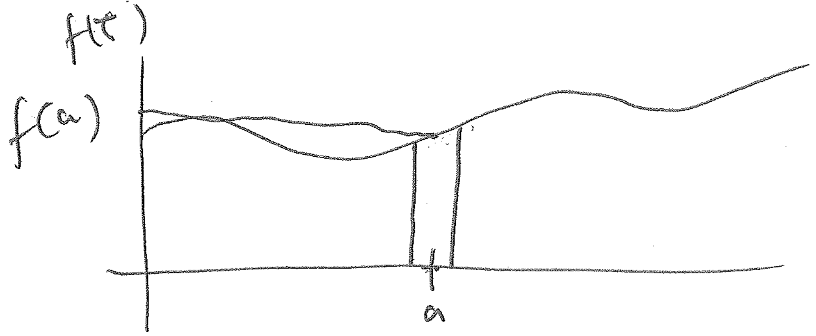
area is always 1 regardless of σ.

- in terms of its action

let $d_\varepsilon(t-a) = \begin{cases} 1/\varepsilon & a - \frac{\varepsilon}{2} < t < a + \frac{\varepsilon}{2} \\ 0 & \text{otherwise} \end{cases}$

$$\int_{-\infty}^{\infty} f(t)\, \underbrace{d_\varepsilon(t-a)}_{\uparrow \text{"action"}}\, dt = \int_{a-\frac{\varepsilon}{2}}^{a+\frac{\varepsilon}{2}} f(t)\, \frac{1}{\varepsilon}\, dt.$$

for $\varepsilon \ll 1$, $f(t) \sim f(a) + O(\varepsilon)$ on the interval $(a - \frac{\varepsilon}{2}, a + \frac{\varepsilon}{2})$



$$\Rightarrow \int_{-\infty}^{\infty} f(t)\, d_\varepsilon(t-a)\, dt \sim f(a) \underbrace{\boxed{\int_{a-\frac{\varepsilon}{2}}^{a+\frac{\varepsilon}{2}} \frac{1}{\varepsilon}\, dt.}}_{1}$$

$$= f(a) + O(+\varepsilon)$$

in the limit $\varepsilon \to 0$ $d_\varepsilon(t-a) \to \delta(t-a)$.

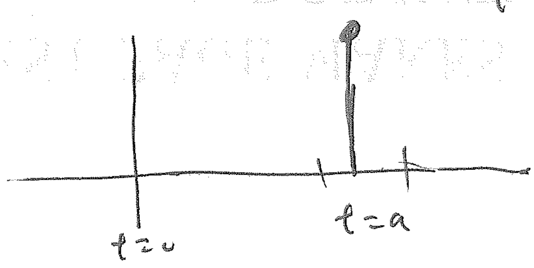
we have by def'n,

$$\int_{-\infty}^{\infty} f(t)\, \delta(t-a)\, dt = f(a)$$

referred to as the action of $\delta(t-a)$ on $f(t)$.

- in terms of measure :

$$\int_I \delta(t-a)\, dt = \begin{cases} 1 & \text{if } a \in I. \\ 0 & \text{if } a \notin I. \end{cases}$$



note: $\displaystyle\int_{-\infty}^{t} \delta(t-a)\,dt = \begin{cases} 0 & t<a \\ 1 & t>a \end{cases}$

$= u(t-a)$.

Laplace transform of δ-fcn:

$$\mathcal{L}(\delta(t-a)) = \int_0^\infty e^{-st}\,\delta(t-a)\,dt = e^{-as}$$

$a>0$.

<u>EX</u> p.319

solve $x'' + 4x = 8\,\delta(t-2\pi)$

$x(0)=3, \quad x'(0)=0$.

[qualitatively, we have pure sinusoidal oscillations for $t<2\pi$ and $t>2\pi$, with only a "kick" at $t=2\pi$]

apply $\mathcal{L}(\cdot)$:

$$s^2 X - s x(0) - \cancel{x'(0)} + 4X = 8 e^{-2\pi s}$$

$$(s^2+4)X = 8 e^{-2\pi s} + 3s$$

$$X(s) = \frac{8e^{-2\pi s}}{s^2+4} + \frac{3s}{s^2+4}$$

$$= 4 e^{-2\pi s} \cdot \frac{2}{s^2+4} + 3 \frac{s}{s^2+4}$$

↑
shift in time.

$$x(t) = 4u(t-2\pi)\sin(2(t-2\pi)) + 3\cos 2t.$$

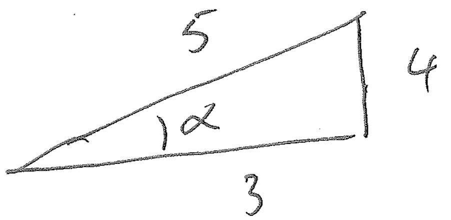
$$x(t) = \begin{cases} 3\cos 2t & 0 < t < 2\pi. \\ 3\cos 2t + 4\sin 2t & t > 2\pi. \end{cases}$$

$$= \begin{cases} 3\cos 2t & 0 < t < 2\pi. \\ 5\cos(2t - 0.9273) & t > 2\pi. \end{cases}$$

note: change in phase and amplitude as a result of the kick.

$$\left[\begin{array}{l} 3\cos 2t + 4\sin 2t = C\cos(2t-\alpha) \\ C = 5, \quad \alpha = \tan^{-1}(4/3) = 0.9273 \end{array}\right]$$

$$x'(t) = \begin{cases} -6\sin 2t & 0 < t < 2\pi \\ -10\sin(2t-\alpha) & t > 2\pi. \end{cases}$$



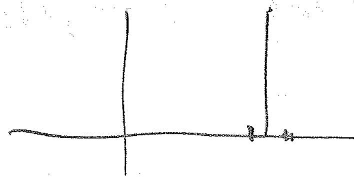
$$x'(2\pi^+) - x'(2\pi^-) = -10\sin(4\pi-\alpha) \cancel{-(-6\sin 4\pi)}$$

$$= -10\sin(4\pi-\alpha) = 10\sin\alpha = 10\cdot\frac{4}{5} = 8$$

why 8?

look at ODE:

$$x'' + 4x = 8\,\delta(t - 2\pi)$$



integrate across:

$$\int_{2\pi^-}^{2\pi^+} x''\,dt + \cancel{\int_{2\pi^-}^{2\pi^+} 4x\,dt} = \int_{2\pi^-}^{2\pi^+} 8\,\delta(t-2\pi)\,dt$$

$$= 8$$

say this is 0

$$x'(2\pi^+) - x'(2\pi^-) = 8.$$

$\int_{2\pi^-}^{2\pi^+} x\,dt = 0$ since x is cont's and finite

if this is not 0, x must behave like a δ-fcn near $t = 2\pi$, but then x'' would be "worse" than a δ-fcn. Since no such fcn on the RHS exists, $\int_{2\pi^-}^{2\pi^+} x\,dt = 0$

note: x cont's
x' discont's
x'' is a δ

x gets progressively worse by differentiation.

In general, the worst behavior on the RHS is always reflected in the highest derivative on the LHS.

to solve without Laplace: define

$$x(t) = \begin{cases} x_l(t) & 0 < t < 2\pi \\ x_r(t) & t > 2\pi \end{cases}$$

$x_l'' + x_l = 0$, $x_l(0) = 3$, $x_l'(0) = 0$.

$x_r'' + x_r = 0$

$x_r(2\pi) = x_l(2\pi)$

$$\underbrace{x_r'(2\pi) = x_l'(2\pi) + 8}_{\text{jump condition}}$$