Last time, we wrote

$mx'' + cx' + kx = f(t)$ as a system

of 2 ODE's

$$\frac{d\vec{v}}{dt} = \underline{\underline{A}}\,\vec{v} + \vec{f}(t) \quad ; \quad \vec{f}(t) = \begin{pmatrix} 0 \\ f(t) \end{pmatrix}$$

$$\underline{\underline{A}} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix}$$

easiest scheme is forward Euler scheme (explicit)

$$\frac{d\vec{v}}{dt} \to \frac{\vec{v}_{i+1} - \vec{v}_i}{\Delta t} \quad , \quad A\vec{v} = A\vec{v}_i$$

where $\vec{v}_i$ is $\vec{v}(t_i)$, $t_i = i\Delta t$.

$\Delta t \ll 1$

$$\vec{v}_{i+1} = \vec{v}_i + \Delta t\left[\underline{\underline{A}}\,\vec{v}_i + \vec{f}(t_i)\right].$$

$i=0$: $\vec{v}_0 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

error is $O(\Delta t)$ so ~~a~~ smaller Δt $\rightarrow$ greater accuracy, but more expensive to compute

there are more accurate schemes (Leapfrog, Runge-Kutta etc) $\rightarrow$ error $O(\Delta t^2)$, $O(\Delta t^4)$ ...

Some definitions and properties of matrices

let $\underline{\underline{A}} = \begin{pmatrix} 3 & 2-i \\ 4+3i & -5+2i \end{pmatrix}$

$\underline{\underline{A}}^T = \begin{pmatrix} 3 & 4+3i \\ 2-i & -5+2i \end{pmatrix}$ (transpose)

$\bar{\underline{\underline{A}}} = \begin{pmatrix} 3 & 2+i \\ 4-3i & -5-2i \end{pmatrix}$ (conjugate)

$\underline{\underline{A}}^* = \begin{pmatrix} 3 & 4-3i \\ 2+i & -5-2i \end{pmatrix}$ conjugate transpose, Hermitian transpose..

scalar multiplication

~~for const~~

$$\alpha \underline{\underline{A}} = \begin{pmatrix} 3\alpha & (2-i)\alpha \\ (4+3i)\alpha & (-5+2i)\alpha \end{pmatrix}$$

if $\underline{\underline{A}}\vec{\xi} = \lambda\vec{\xi}$, then $\boxed{\alpha \underline{\underline{A}}}\underline{\vec{\xi}} = \boxed{\alpha\lambda}\underline{\vec{\xi}}$

so eve's get scaled by α, evr's unchanged.

differentiation

- let $(\underline{\underline{A}})_{ij}$ ~~denote~~ denote the ij^{th} component of $\underline{\underline{A}}$

then $\left(\frac{d}{dt}\underline{\underline{A}}\right)_{ij} = \frac{d}{dt}(\underline{\underline{A}})_{ij}$ where $\underline{\underline{A}} = \underline{\underline{A}}(t)$

eg. $\frac{d}{dt}\begin{pmatrix} 3t & 5 \\ e^{2t} & \sin t \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 2e^{2t} & \cos t \end{pmatrix}$.

- also $\frac{d}{dt}\underline{\underline{C}}\,\underline{\underline{A}} = \underline{\underline{C}}\frac{d\underline{\underline{A}}}{dt} \neq \frac{d\underline{\underline{A}}}{dt}\underline{\underline{C}}$ $\quad \underline{\underline{C}}$ constant matrix

- $\frac{d}{dt}(\underline{\underline{A}} + \underline{\underline{B}}) = \frac{d\underline{\underline{A}}}{dt} + \frac{d\underline{\underline{B}}}{dt}$.

- $\frac{d}{dt}\underline{\underline{A}}(t)\underline{\underline{B}}(t) = \underbrace{\frac{d\underline{\underline{A}}}{dt}\underline{\underline{B}} + \underline{\underline{A}}\frac{d\underline{\underline{B}}}{dt}}_{\text{note the order.}}$

- the $n \times n$ system $\frac{d\vec{v}}{dt} = \underline{\underline{A}}(t)\vec{v} + \vec{f}(t)$ (*)

is homogeneous if $\vec{f} \equiv \vec{0}$, and non-homog. otherwise.

<u>Existence and uniqueness</u>

suppose $(\underline{\underline{A}}(t))_{ij}$ and $\vec{f}(t)$ are cont's for $1 \le i, j \le n$ on interval I containing point $t=a$, then given $b_1, \ldots, b_n$, (*) has unique solution satisfying

$$\vec{v}(a) = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

proof: simple extension of theorem for scalar version

principle of superposition

let $\vec{v}_1, \ldots, \vec{v}_n$ be solns of (*) with $\vec{f}(t) = \vec{0}$

then if $c_1, \ldots, c_n$ are constants,

then

$$\vec{v} = c_1 \vec{v}_1 + \cdots + c_n \vec{v}$$ is also a solution.

proof:
$$\frac{d\vec{v}_i}{dt} = \underline{\underline{A}}\,\vec{v}_i$$

so
$$\frac{d\vec{v}}{dt} = c_1 \frac{d\vec{v}_1}{dt} + \cdots + c_n \frac{d\vec{v}_n}{dt}$$
$$= c_1 \underline{\underline{A}}\,\vec{v}_1 + \cdots + c_n \underline{\underline{A}}\,\vec{v}_n$$
$$= \underline{\underline{A}}\left[c_1 \vec{v}_1 + \cdots + c_n \vec{v}\right]$$
$$= \underline{\underline{A}}\,\vec{v}. \quad \boxtimes$$

linear indep.

suppose $\vec{v}_1, \ldots, \vec{v}_n$ are soln's of (*) with $\vec{f} = 0$ and $\underline{\underline{A}}(t)$ cont's. Then $\vec{v}_i$ are l.d. if there exists constants $c_1, \dots, c_n$ not all 0 s.t. $c_1 \vec{v}_1(t) + \dots + c_n \vec{v}_n(t) = 0$ for all t. equivalently, $\vec{v}_i$ l.d. if

$$W = \det(\vec{v}_1 \; \vec{v}_2 \; \cdots \; \vec{v}_n) = 0 \text{ for all } t.$$

$\vec{v}_i$ l.i. if $W \neq 0$ for any t (W is either always 0 or never 0)

General solution of $\dfrac{d\vec{v}}{dt} = \underset{n \times n}{\underline{A}(t)} \underset{n \times 1}{\vec{v}}$ $\quad \vec{v}(t_0) = \vec{v}_0$

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let $\underline{A}(t)$ be cont's for all t. and

$\dfrac{d\vec{v}_i}{dt} = A\vec{v}_i$ for $i = 1, \dots, n$, $\vec{v}_i$ l.i.

then the general solution $\vec{v}$ may be written

$$\vec{v} = c_1 \vec{v}_1(t) + \dots + c_n \vec{v}(t)$$

where c_i are given by

$$\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \underbrace{\left(\vec{v}_1(t_0) \; \vec{v}_2(t_0) \cdots \vec{v}_n(t_0) \right)}_{\text{invertible because } \vec{v}_i \text{ l.i.}}^{-1} \vec{v}_0.$$

by existence and uniqueness thm, this is the only soln.

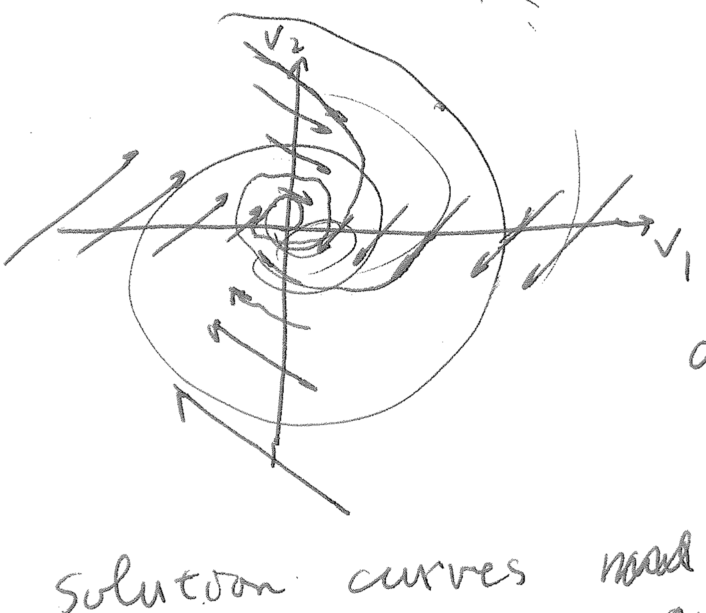
phase portrait / direction field

plots the direction of $\underline{\underline{A}}\vec{v}$ at various points $\vec{v}$. for a 2x2 system, we can visualize this

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \qquad \underline{\underline{A}} = \begin{pmatrix} -1/2 & 1 \\ -1 & -1/2 \end{pmatrix}.$$

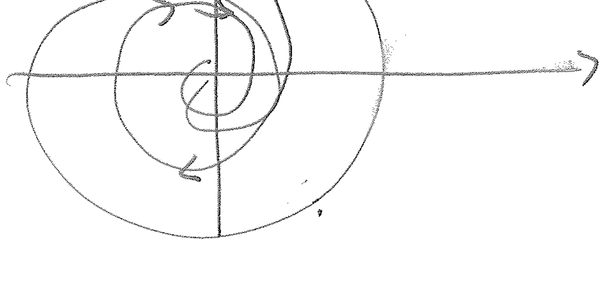
$$\vec{v} = \begin{pmatrix} b \\ 0 \end{pmatrix} \Rightarrow A\vec{v} = b\begin{pmatrix} -1/2 \\ -1 \end{pmatrix} \quad \swarrow$$

$$\vec{v} = \begin{pmatrix} 0 \\ b \end{pmatrix} = b\begin{pmatrix} 1 \\ -1/2 \end{pmatrix}. \quad \searrow$$



direction field

solution curves must all be tangent to the arrows.



phase portrait.

Ex

1) solve $\frac{d\vec{v}}{dt} = \underbrace{\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}}_{\underline{\underline{A}}}$ and sketch the phase portrait.

guess $\vec{v} = \vec{\xi} e^{\lambda t}$

find λ and $\vec{\xi}$

$$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0$$

$$\det \begin{pmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{pmatrix} = 0.$$

$$(1-\lambda)^2 - 4 = 0 \qquad (\text{C.E.})$$

$$1-\lambda = \pm 2 \Rightarrow \lambda_+ = 3, \; \lambda_- = -1$$

find $\vec{\xi}^+$ $(\lambda = \lambda_+)$

$$\begin{pmatrix} 1-3 & 1 \\ 4 & 1-3 \end{pmatrix} \begin{pmatrix} \xi_1^+ \\ \xi_2^+ \end{pmatrix} = 0$$

$$-2\xi_1^+ + \xi_2^+ = 0 \qquad \text{pick } \xi_1^+ = 1$$

$$\xi_2^+ = 2.$$

$$\vec{\xi}^+ = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$\vec{\xi}^{-}$ $(a = \lambda_{-}^{\bullet})$

$$\begin{pmatrix} 1-(-1) & 1 \\ 4 & 1-(-1) \end{pmatrix} \begin{pmatrix} \xi_1^- \\ \xi_2^- \end{pmatrix} = 0.$$

$$2\xi_1^- + \xi_2^- = 0 \qquad \xi_1^- = 1, \ \xi_2^- = -2.$$

$$\vec{\xi}^{-} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

$$\vec{v} = c_+ \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_- \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}.$$

if given IC's $\vec{v}(0) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$,

$$c_+ \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_- \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} c_+ \\ c_- \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

$$\begin{pmatrix} c_+ \\ c_- \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}^{-1} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

note: if $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \gamma \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, then $c_+ = \gamma$, $c_- = 0$

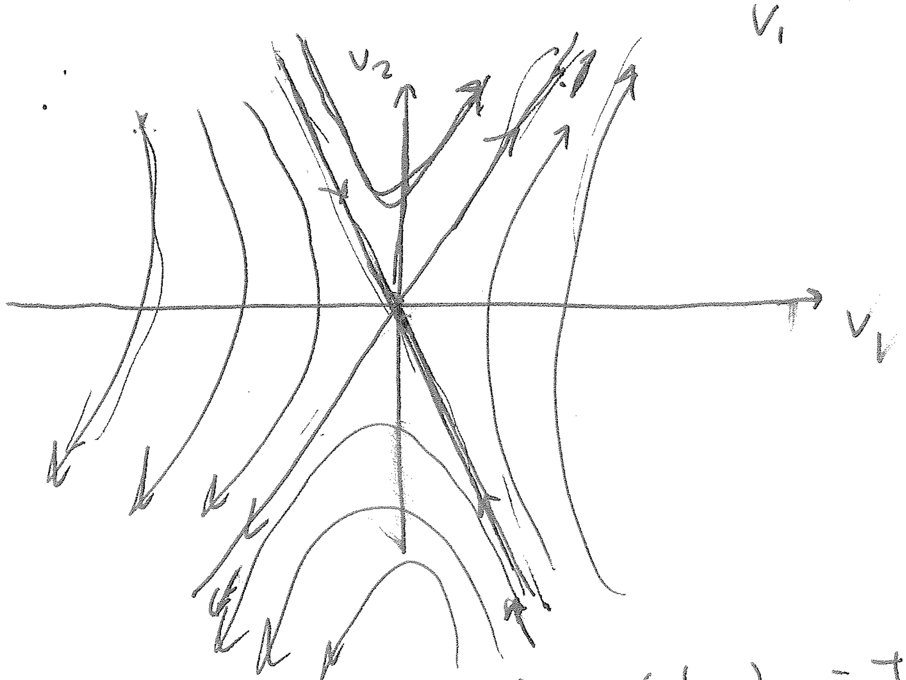
if $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \gamma \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ then $c_+ = 0$, $c_- = \gamma$.

<u>phase portrait</u> (v_2 vs v_1)

first draw trajectories corresponding to $c_+ = 1$, $c_- = 0$. and $c_+ = 0$, $c_- = 1$.

if $c_+ = 1$, $c_- = 0$, $\vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}$

$v_2 = 2e^{3t}$, $v_1 = e^{3t}$ $\quad \frac{v_2}{v_1} = 2 \quad v_2 = 2v_1$



$\frac{d\vec{v}}{dt} = A\vec{v}$.

if $c_+ = 0$, $c_- = 1$, $\vec{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$.

$v_2 = -2v_1$

$\vec{v} = c_+ \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_- \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$.

origin is called a saddle point, eve's opposite signs, origin is <u>unstable</u>.

EX solve $\frac{d\vec{v}}{dt} = \underbrace{\begin{pmatrix} 1 & 1 \\ -3 & 5 \end{pmatrix}}_{\underline{A}} \vec{v}$ and plot phase portrait.

$$\vec{v} = \vec{\xi} e^{\lambda t}.$$

$$\det(\underline{A} - \lambda I) = 0 \quad \det\begin{pmatrix} 1-\lambda & 1 \\ -3 & 5-\lambda \end{pmatrix} \xi = 0.$$

$$(1-\lambda)(5-\lambda) + 3 = 0 \Rightarrow \lambda_+ = 4, \ \lambda_- = 2$$

$$\vec{\xi}^+ : \begin{pmatrix} 1-4 & 1 \\ -3 & 5-4 \end{pmatrix} \begin{pmatrix} \xi_1^+ \\ \xi_2^+ \end{pmatrix} = 0.$$

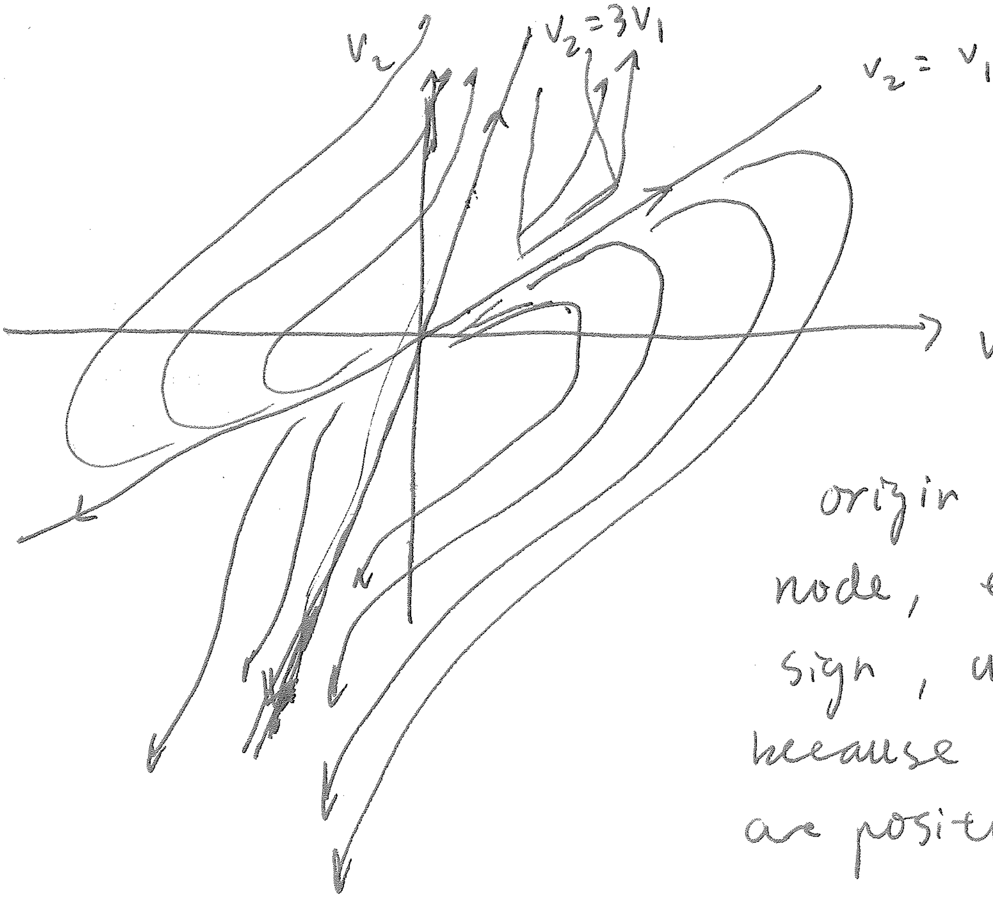
$$-3\xi_1^+ + \xi_2^+ = 0. \quad \xi_1^+ = 1, \ \xi_2^+ = 3.$$

$$\vec{\xi}^+ = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

$$\vec{\xi}^- : \begin{pmatrix} 1-2 & 1 \\ \text{—} & \text{—} \end{pmatrix} \begin{pmatrix} \xi_1^- \\ \xi_2^- \end{pmatrix} = 0.$$

$$-\xi_1^- + \xi_2^- = 0 \rightarrow \vec{\xi}^- = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

then $\vec{v} = c_+ \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{4t} + c_- \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$

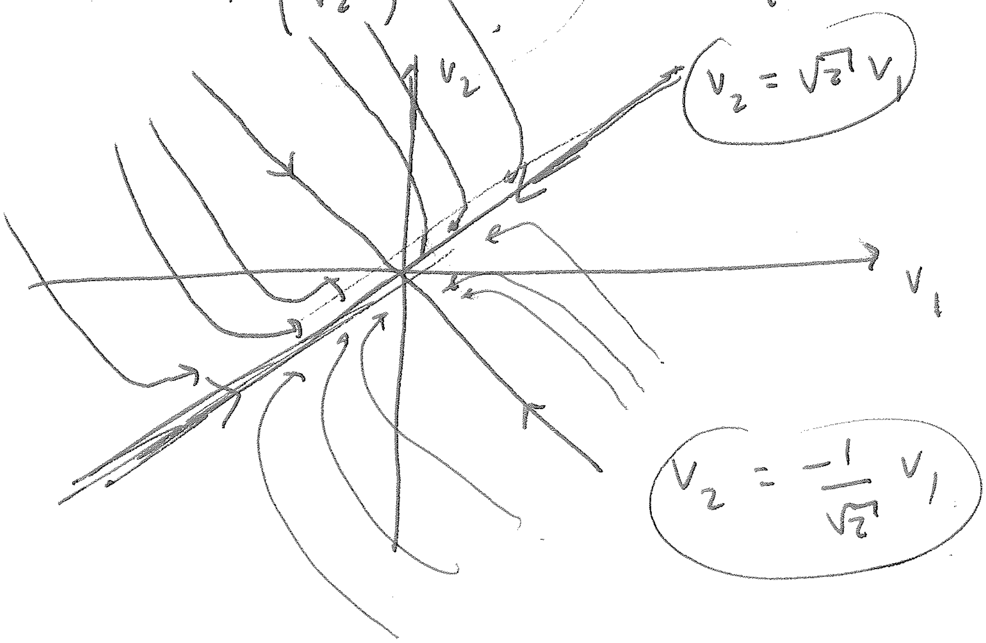


origin is a node, eve's same sign, unstable because eve's are positive real.

trajectories never cross.

<u>Ex</u> plot the phase portrait of

$$\vec{v} = c_1 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t}.$$



node, eve same sign, stable.

Ex find the general solution of

$$\frac{d\vec{v}}{dt} = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix} \vec{v}$$

and write in terms of sines and cosines.
and plot phase portrait.

$$\det \begin{pmatrix} -\frac{1}{2} - \lambda & 1 \\ -1 & -\frac{1}{2} - \lambda \end{pmatrix} = 0$$

$$\left(\lambda + \frac{1}{2}\right)^2 + 1 = 0 .$$

$$\lambda = -\frac{1}{2} \pm i$$

find $\vec{\xi}^{+}$:

$$\begin{pmatrix} -\frac{1}{2} - \left(-\frac{1}{2} + i\right) & 1 \\ - & - \end{pmatrix} \begin{pmatrix} \xi_1^{+} \\ \xi_2^{+} \end{pmatrix} = 0.$$

$$-i\,\xi_1^{+} + \xi_2^{+} = 0. \qquad \xi_1^{+} = 1 , \quad \xi_2^{+} = i$$

$$\vec{\xi}^{+} = \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

$\vec{\xi}^{-}$:

$$\begin{pmatrix} -\frac{1}{2} - \left(-\frac{1}{2} - i\right) & 1 \\ - & - \end{pmatrix} \begin{pmatrix} \xi_1^{-} \\ \xi_2^{-} \end{pmatrix} = 0.$$

$$i\xi_1^{-} + \xi_2^{-} = 0 \Rightarrow \vec{\xi}^{-} = \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

$$\vec{v} = c_1 \begin{pmatrix} 1 \\ i \end{pmatrix} e^{\left(-\frac{1}{2} + i\right)t} + c_2 \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{\left(-\frac{1}{2} - i\right)t}.$$

$$= e^{-\frac{1}{2}t}\left[c_1 \begin{pmatrix} 1 \\ i \end{pmatrix} (\cos t + i\sin t) + c_2 \begin{pmatrix} 1 \\ -i \end{pmatrix} (\cos t - i\sin t)\right]$$

$$= e^{-\frac{1}{2}t}\left[c_1 \begin{pmatrix} \cos t + i\sin t \\ -\sin t + i\cos t \end{pmatrix} + c_2 \begin{pmatrix} \cos t - i\sin t \\ -\sin t - i\cos t \end{pmatrix}\right]$$

$$= e^{-\frac{1}{2}t}\left[\underbrace{(c_1 + c_2)}_{a} \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + \underbrace{i(c_1 - c_2)}_{b} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}\right].$$

$$= e^{-\frac{1}{2}t}\left[a \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + b \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}\right].$$

one l.i. solution

$\underbrace{\qquad\qquad}$ + $\underbrace{\qquad\qquad}$ one l.i. solution can also take real and imag. ~~of~~ parts

of $\vec{\xi}_+ e^{\lambda_+ t}$:

$$\vec{\xi}_+ e^{\lambda_+ t} = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{-\frac{1}{2}t} (\cos t + i \sin t)$$

$$= e^{-\frac{1}{2}t} \begin{pmatrix} \cos t + i \sin t \\ -\sin t + i \cos t \end{pmatrix}.$$

$$= \underbrace{e^{-\frac{1}{2}t} \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}}_{\text{real part}} + i \underbrace{e^{-\frac{1}{2}t} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}}_{\text{imag}}.$$

$\vec{v} = a \qquad + b$