~~a~~ last time

$$\frac{d\vec{v}}{dt} = \begin{pmatrix} -1/2 & 1 \\ -1 & -\frac{1}{2} \end{pmatrix} \vec{v}.$$

wrote solution in terms of sines and cosines

$$\lambda = -\frac{1}{2} \pm i$$

$$\vec{\xi} = \begin{pmatrix} 1 \\ -i \end{pmatrix} \quad , \quad \begin{pmatrix} 1 \\ +i \end{pmatrix}$$

$\begin{pmatrix} 1 \\ -i \end{pmatrix} e^{\left(-\frac{1}{2}+i\right)t} \longrightarrow$ real / imag of this

$$\hookrightarrow \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} e^{-\frac{1}{2}t} + i \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} e^{-\frac{1}{2}t}.$$

$$\vec{v} = c_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} e^{-\frac{1}{2}t} + c_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} e^{-\frac{1}{2}t}.$$

say $c_2 = 0$.

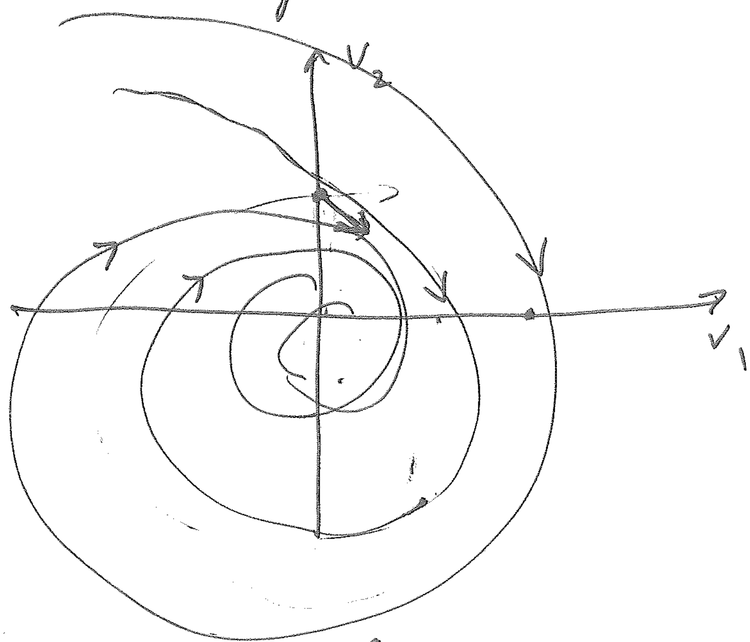
draw phase portrait (take $c_2 = 0$).

then

$$v_1 = \cos t \, e^{-\frac{1}{2}t}, \quad v_2 = -\sin t \, e^{-\frac{1}{2}t}.$$

$$v_1^2 + v_2^2 = e^{-t}\left[\cos^2 t + \sin^2 t\right] = e^{-t}.$$

in $v_1 - v_2$ space, this is a circle with decreasing radius. (inward spiral)



direction?

pick point

$\vec{v} = (0, 1)$

$A\vec{v} = (1, -\frac{1}{2})$

$\Rightarrow$ clockwise

[origin is a spiral, stable (real(λ) < 0)].

↑ trajectories never cross.

Ex

$$\frac{d\vec{v}}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \vec{v}$$

write soln in terms of sines and cosines find eve's, evr's:

$$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0.$$

$$\det\begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix} = 0 \quad \Rightarrow \quad \lambda^2 + 1 = 0$$

$$\lambda = \pm i$$

evr's (only need one).

$$\begin{pmatrix} -i & 1 \\ - & - \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0.$$

$$-i\xi_1 + \xi_2 = 0 \qquad \xi_1 = 1, \quad \xi_2 = i$$

then $\vec{\xi} = \begin{pmatrix} 1 \\ i \end{pmatrix}$.

then just need real/imag parts of

$$\begin{pmatrix} 1 \\ i \end{pmatrix} e^{it} = \begin{pmatrix} 1 \\ i \end{pmatrix} (\cos t + i \sin t)$$

$$= \begin{pmatrix} \cos t + i \sin t \\ -\sin t + i \cos t \end{pmatrix} = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + i \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}.$$

so 2 l.i. soln's are.

$$\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}, \quad \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

$\lambda = -i, \quad \vec{\xi} = \begin{pmatrix} 1 \\ -i \end{pmatrix}.$

$$\hookrightarrow \vec{v} = c_1 \begin{pmatrix} 1 \\ i \end{pmatrix} e^{it} + c_2 \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{-it}$$

$$= c_1 \begin{pmatrix} 1 \\ i \end{pmatrix} (\cos t + i \sin t) + c_2 \begin{pmatrix} 1 \\ -i \end{pmatrix} (\cos t - i \sin t)$$

$$= \begin{pmatrix} c_1 \cos t + i c_1 \sin t + c_2 \cos t - i c_2 \sin t \\ i c_1 \cos t - c_1 \sin t - i c_2 \cos t - c_2 \sin t \end{pmatrix}.$$

$$= \begin{pmatrix} \overset{a}{(c_1 + c_2)} \cos t + i (\overset{b}{(c_1 - c_2)}) \sin t. \\ -(c_1 + c_2) \sin t + i (c_1 - c_2) \cos t \end{pmatrix}.$$

$$= a \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + \underbrace{ib}_{\tilde{b}} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}.$$

so equivalent.

so we have

$$\vec{v} = c_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}.$$

if given

$$\vec{v}(0) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

$$\vec{v}(0) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

$$\Rightarrow c_1 = \alpha, \quad c_2 = \beta.$$

phase diagram:

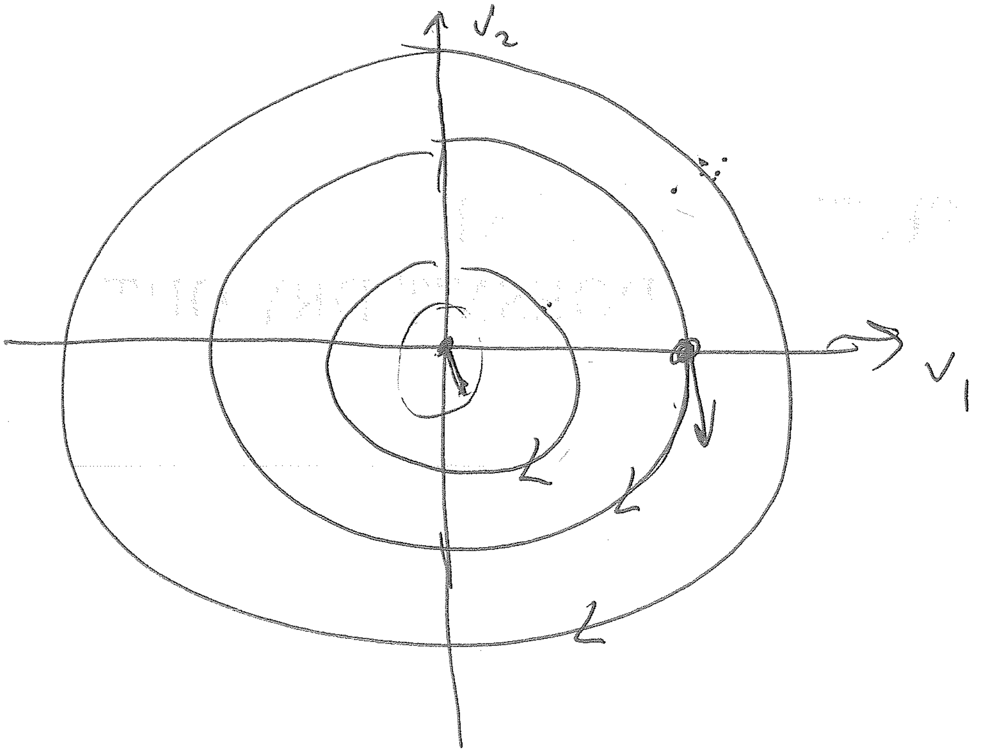
$$v_1^2 + v_2^2 = (c_1 \cos t + c_2 \sin t)^2 + (-c_1 \sin t + c_2 \cos t)^2$$

$$= c_1^2 \cos^2 t + \cancel{2 c_1 c_2 \cos t \sin t} + c_2^2 \sin^2 t$$

$$+ c_1^2 \sin^2 t - \cancel{2 c_1 c_2 \sin t \cos t} + c_2^2 \cos^2 t.$$

$$= (c_1^2 + c_2^2) \cos^2 t + (c_1^2 + c_2^2) \sin^2 t.$$

$$= (c_1^2 + c_2^2)$$



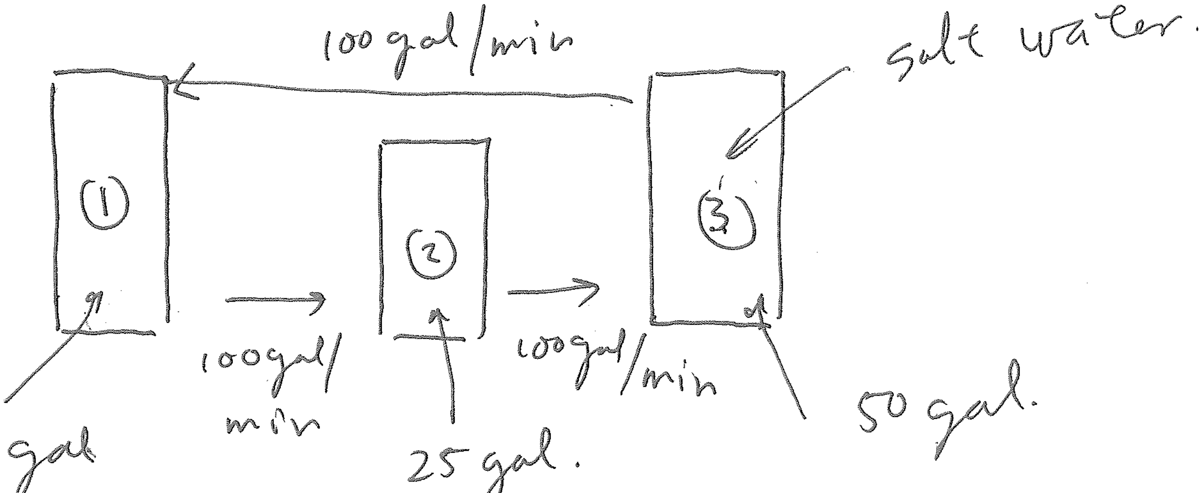
direction? clockwise.

say pick $\vec{v} = \begin{pmatrix}1\\0\end{pmatrix}$, $A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\-1\end{pmatrix}$.

classify the origin: origin is a <u>center</u>

neutrally stable (real(λ) = 0)

<u>EX</u> (p.376 in book).



find amounts $x_1(t)$, $x_2(t)$, $x_3(t)$ of salt at time t in each tank.

$$\frac{dx_1}{dt} = (100 \text{ gal/min}) \left(\frac{x_3(t) \text{ kg}}{50 \text{ gal}} \right) - (100 \text{ gal/min}) \left(\frac{x_1(t)}{50} \right).$$

$$\frac{dx_2}{dt} = 100 \cdot \frac{x_1}{50} - 100 \frac{x_2}{25}$$

$$\frac{dx_3}{dt} = 100 \cdot \frac{x_2}{25} - 100 \cdot \frac{x_3}{50}.$$

$$\underset{\uparrow \vec{x}}{\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} = \underset{\uparrow \underset{=}{A}}{\begin{pmatrix} -2 & 0 & 2 \\ 2 & -4 & 0 \\ 0 & 4 & -2 \end{pmatrix}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

guess:

$$\vec{x} = \vec{\xi} e^{\lambda t}.$$

$$\det \begin{pmatrix} -2-\lambda & 0 & 2 \\ 2 & -4-\lambda & 0 \\ 0 & 4 & -2-\lambda \end{pmatrix} = 0.$$

$$(-2-\lambda)(-4-\lambda)(-2-\lambda) + 2(2)(4) = 0.$$

$$\lambda = 0, \quad -4+2i, \quad -4-2i$$

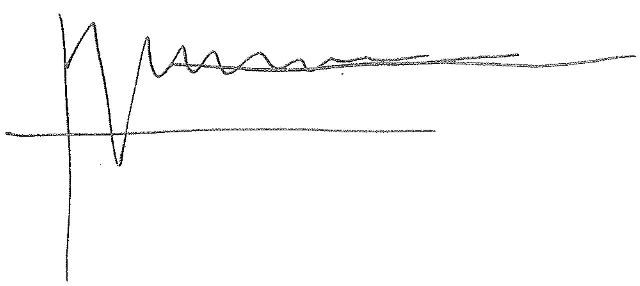
$$\vec{\xi} \qquad \vec{\xi}$$

$\underline{\lambda = 0}$

$$\begin{pmatrix} -2 & 0 & 2 \\ 2 & -4 & 0 \\ - & - & - \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = 0.$$

$$\left. \begin{aligned} -2\xi_1 + 2\xi_3 &= 0. \\ 2\xi_1 - 4\xi_2 &= 0. \end{aligned} \right\} \qquad \text{Pick } \xi_1 = 2, \quad \xi_3 = 2, \quad \xi_2 = 1.$$

$$\vec{\xi} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}.$$



A36

$\lambda = -4 + 2i \qquad -2 - (-4 + 2i)$

$\qquad\qquad\qquad\qquad 2 - 2i$

$$\begin{pmatrix} 2-2i & 0 & 2 \\ 2 & -2i & 0 \\ - & - & - \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = 0.$$

$\underline{\underline{A}} - \lambda \underline{\underline{I}}$

$(2-2i)\xi_1 + 2\xi_3 = 0.$

$2\xi_1 - 2i\xi_2 = 0. \qquad 2 - 2i\xi_2 = 0. \quad \xi_2 = -i$

$\xi_1 = 1 \Rightarrow \xi_2 = -i \Rightarrow \xi_3 = -(1-i).$

$\vec{\xi} = \begin{pmatrix} 1 \\ -i \\ -1+i \end{pmatrix}$ pick off real / imag. of

$$\begin{pmatrix} 1 \\ -i \\ -1+i \end{pmatrix} e^{(-4+2i)t} = e^{-4t} \begin{pmatrix} 1 \\ -i \\ -1+i \end{pmatrix} (\cos 2t + i \sin 2t)$$

$$= e^{-4t} \begin{pmatrix} \cos 2t + i \sin 2t \\ +\sin 2t - i \cos 2t \\ -\cos 2t - \sin 2t + i \cos 2t - i \sin 2t \end{pmatrix}.$$

$$= e^{-4t} \left[\begin{pmatrix} \cos 2t \\ +\sin 2t \\ -\cos 2t - \sin 2t. \end{pmatrix} + i \begin{pmatrix} \sin 2t \\ -\cos 2t \\ \cos 2t - \sin 2t \end{pmatrix} \right].$$

so we have

$$\vec{v} = c_1 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \cancel{e^{0t}} + e^{-4t} \left[c_2 \begin{pmatrix} \cos 2t \\ \sin 2t \\ -\cos 2t - \sin 2t \end{pmatrix} + c_3 \begin{pmatrix} \sin 2t \\ -\cos 2t \\ \cos 2t - \sin 2t \end{pmatrix} \right].$$

notice $x_1 + x_2 + x_3 = 5c_1$

$\lim_{t\to\infty} x_1 = 2c_1$, $\lim_{t\to\infty} x_2 = c_1$, $\lim_{t\to\infty} x_3 = 2c_1$.

40%   20%   40%

Repeated eve's (5.6) (skipped 5.5).

two possibility:

① eve's associated with distinct linearly indep. evr's.

e.g. $\underline{\underline{A}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\lambda_1 = \lambda_2 = 1$.

$\vec{\xi}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{\xi}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\underline{\underline{A}} - \underline{\underline{I}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, ② only one evr

e.g. $\underline{\underline{A}} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$.

$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = \begin{pmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{pmatrix} = 0.$

$(1-\lambda)(3-\lambda) + 1 = 0.$

$\lambda^2 - 4\lambda + 3 + 1 = 0.$

$\lambda^2 - 4\lambda + 4 = 0 \Rightarrow (\lambda - 2)^2 = 0$

$\lambda = 2, 2.$

find evr:

$\begin{pmatrix} -1 & -1 \\ \cancel{1} & \cancel{1} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \Rightarrow \vec{\xi} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$

only 1 evr.

<u>EX</u> $\dfrac{d\vec{v}}{dt} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \vec{v}.$

find evl :

$$\det \begin{pmatrix} \alpha-\lambda & 0 \\ 0 & \alpha-\lambda \end{pmatrix} = 0 \qquad (\alpha-\lambda)^2 = 0$$

$$\lambda = \alpha, \alpha.$$

find evr :

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0.$$

$$0\,\xi_1 + 0\,\xi_2 = 0.$$

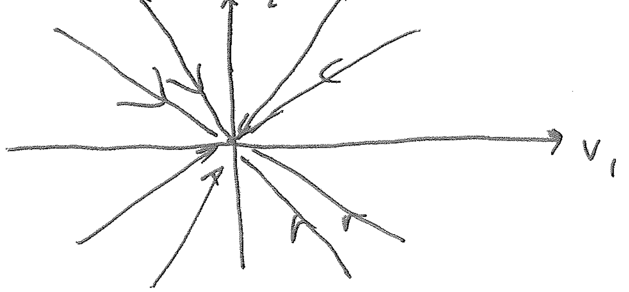
so choose $\vec{\xi}^+$ and $\vec{\xi}^-$ to be any two l.i. vectors. easiest is

$$\vec{\xi}^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \vec{\xi}^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

$$\left[\text{ or } \vec{\xi}^+ = \begin{pmatrix} 1 \\ 7 \end{pmatrix}, \quad \vec{\xi}^- = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \right].$$

so $$\vec{v} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{\alpha t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{\alpha t}.$$

if $\alpha < 0$



$v_1 = c_1 e^{\alpha t}$

$v_2 = c_2 e^{\alpha t}$

$v_2 / v_1 = \dfrac{c_2}{c_1}$

origin is stable, proper node

↑ classification.

↑

real(λ) < 0.

If $\alpha > 0$, then arrow reverse, origin unstable.

what if we have only one l. i. evr?

recall from second order equation,

$$m x'' + b x' + k x = 0.$$

with repeated roots $r_1 = r_2 = \lambda$.

we have

$$x(t) = e^{\lambda t}(c_1 + c_2 t).$$

$$x'(t) = \lambda e^{\lambda t}(c_1 + c_2 t) + e^{\lambda t} c_2$$

so

$$\begin{pmatrix} x \\ x' \end{pmatrix} = e^{\lambda t}\begin{pmatrix} c_1 + c_2 t. \\ c_1\lambda + c_2 + \lambda c_2 t \end{pmatrix}.$$

$$= \begin{pmatrix} c_1 \\ \lambda c_1 + c_2 \end{pmatrix} e^{\lambda t} + t \begin{pmatrix} c_2 \\ \lambda c_2 \end{pmatrix} e^{\lambda t}.$$

recall from p. A10, that $\vec{\xi} = \begin{pmatrix} 1 \\ \lambda \end{pmatrix}$

is an evr of the corresponding 2x2

linear system

$$\left[\quad \underset{=}{A} \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix} \rightarrow A - \lambda I = \begin{pmatrix} -\lambda & 1 \\ - & - \end{pmatrix}. \right.$$

$$\text{so} \quad -\lambda \xi_1 + \xi_2 = 0 \Rightarrow \xi_1 = 1, \ \xi_2 = \lambda. \quad \Big]$$

so in fact,

$$\begin{pmatrix} x \\ x' \end{pmatrix} = c_2 \vec{\xi} \, t e^{\lambda t} + \vec{\eta} \, e^{\lambda t}.$$

↑ evr corresponding to λ ↑ some vector to be found.

this suggest: for the linear system

$$\frac{d\vec{v}}{dt} = \underset{=}{A} \vec{v}$$

where $\underset{=}{A}$ is 2x2, has repeated eve λ with corresponding evr $\vec{\xi}$, the two l.i. solutions have the form

$\vec{v}_1 = \vec{\xi} e^{\lambda t}$. $\quad \vec{v}_2 = \vec{\xi} t e^{\lambda t} + \vec{\eta} e^{\lambda t}$.

need to find $\vec{\eta}$.

<u>Ex</u>

$$\frac{d\vec{v}}{dt} = \begin{pmatrix} -2 & -1 \\ 1 & -4 \end{pmatrix} \vec{v}.$$

$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0 \Rightarrow \lambda^2 + 6\lambda + 9 = 0.$

$(\lambda+3)^2 = 0 \quad \Rightarrow \lambda = -3, -3.$

<u>evr</u> :

$$\begin{pmatrix} -2+3 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ \_ & \_ \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0.$$

$\vec{\xi} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (only 1 evr).

so one solution is

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t}.$$

the other one has the form

$$\vec{v} = \cancel{\vec{\xi} t e^{-3t}} \, \vec{\xi} t e^{\lambda t} + \vec{\eta} \, e^{\lambda t}.$$

find this.

$$\frac{d\vec{v}}{dt} = \vec{\xi}\left[e^{\lambda t} + \lambda t e^{\lambda t}\right] + \lambda \vec{\eta} e^{\lambda t}.$$

$$= A\left[\vec{\xi} t e^{\lambda t} + \vec{\eta} e^{\lambda t}\right].$$

$$\vec{\xi} \cancel{\boxed{+ \lambda \vec{\xi} t}} + \lambda \vec{\eta} = \cancel{\boxed{A \vec{\xi} t}} + A \vec{\eta}.$$

$A\vec{\xi} = \lambda \vec{\xi}$.

therefore,

$$(A - \lambda \underline{\underline{I}})\,\vec{\eta} = \vec{\xi}.$$

1. if $\vec{\eta} = 0$
$\cancel{A - \lambda I} \Rightarrow \vec{\xi} = 0$
2. $\vec{\xi}$ must be the ev for the t-terms to cancel.