last time

$$\frac{d\vec{v}}{dt} = \begin{pmatrix} -2 & -1 \\ 1 & -4 \end{pmatrix} \vec{v}.$$

$\lambda^2 + 6\lambda + 9 = 0 \Rightarrow \lambda = -3, -3.$

(repeated).

only one corresponding eigenvector

$$\vec{\xi} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

so one solution is

$$\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t}.$$

the other has the form.

$$\vec{v} = \vec{\xi}\, t e^{-3t} + \vec{\eta}\, e^{-3t}.$$ for

some $\vec{\eta}$ to be found.

$$\vec{v} = \vec{\xi}\, t e^{\lambda t} + \vec{\eta}\, e^{\lambda t}.$$

$$\frac{d\vec{v}}{dt} = \vec{\xi}\left[e^{\lambda t} + \lambda t e^{\lambda t}\right] + \vec{\eta}\lambda e^{\lambda t} = A\vec{v}.$$

so

$$\vec{\xi} + \cancel{\boxed{\lambda \vec{\xi} t}} + \lambda \vec{\eta} = \cancel{\boxed{A\vec{\xi} t}} + A\vec{\eta}.$$

since $A\vec{\xi} = \lambda\vec{\xi}$

$$(\underline{\underline{A}} - \lambda \underline{\underline{I}})\vec{\eta} = \vec{\xi}.$$

$\vec{\eta}$ is called the generalized evr.

$$\underbrace{\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}}_{\text{singular matrix}} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

in column space of matrix on LHS.

$\vec{\eta}$ is non-unique. but exists.

need only look at one row.

$\eta_1 - \eta_2 = 1$. let η_2 be arbitrary.

$\eta_1 = 1 + \eta_2$.

$$\vec{\eta} = \begin{pmatrix} 1+\eta_2 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \eta_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

so the second l.i. soln is

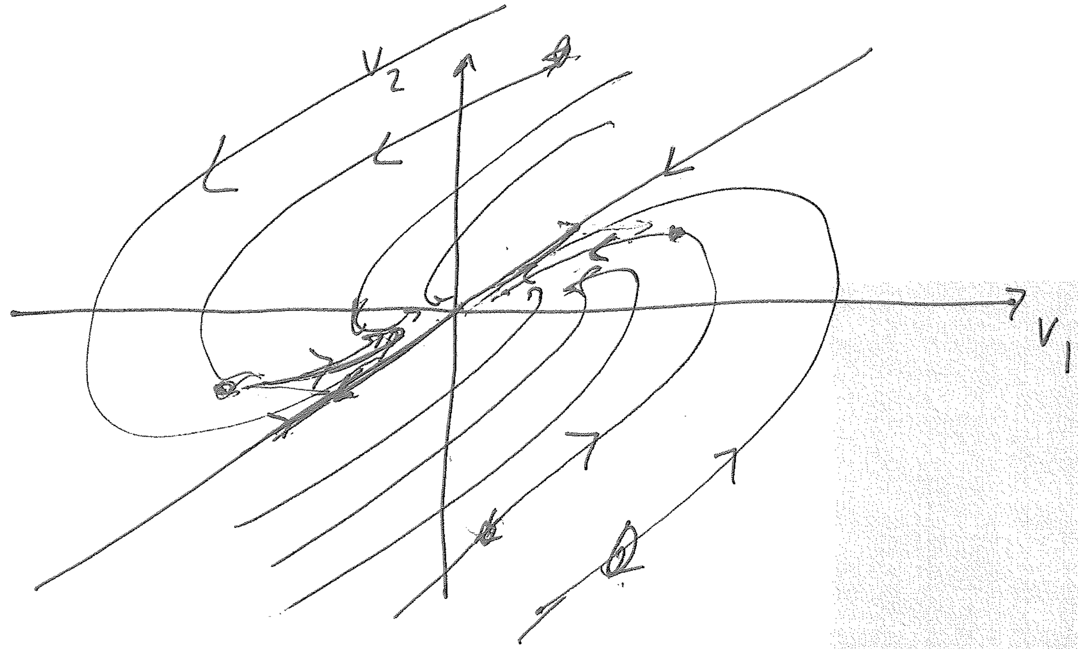
$$\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-3t} + \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \eta_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right] e^{-3t}.$$

$\eta_2 = 0$ wlog.

$$\boxed{\vec{v} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t} + c_2 e^{-3t} \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right].}$$

phase plot:

① draw $\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t}$ $(c_2 = 0)$



note: as $t \to \pm\infty$, trajectories become parallel to $\begin{pmatrix}1\\1\end{pmatrix}$.

(2): plot $c_1\begin{pmatrix}1\\1\end{pmatrix} + c_2\begin{pmatrix}1\\0\end{pmatrix}$ for $c_2 > 0$.

plot $c_1\begin{pmatrix}1\\1\end{pmatrix} + c_2\begin{pmatrix}1\\0\end{pmatrix}$ for $c_2 < 0$.

classify origin: improper node.

stability: stable.

↑ $\text{real}(\lambda) < 0$.

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for 3×3 system with 3 repeated eve's and only one evr, (defect of 2)

eigenpair is $\lambda, \vec{\xi}$

we have

$$\vec{v}_1 = \vec{\xi} e^{\lambda t}.$$

$$\vec{v}_2 = (\vec{\xi} t + \vec{\eta}) e^{\lambda t}.$$

$$\vec{v}_3 = \left(\vec{\xi} \frac{t^2}{2} + \vec{\eta} t + \vec{\zeta} \right) e^{\lambda t}.$$

$$\cancel{\frac{d\vec{v}_3}{dt} = } \quad \frac{d\vec{v}_2}{dt} = A \vec{v}_2 \Rightarrow (A - \lambda I)\vec{\eta} = \vec{\xi}.$$

$$\frac{d\vec{v}_3}{dt} = A \vec{v}_3$$

$$\left[\cancel{\vec{\xi} t} + \underline{\vec{\eta}} \right] e^{\lambda t} + \lambda \left[\cancel{\vec{\xi} \frac{t^2}{2}} \cancel{+ \vec{\eta} t} + \underline{\vec{\zeta}} \right].$$

$$= \cancel{A \vec{\xi} \frac{t^2}{2}} \cancel{+ A\vec{\eta}} t + \underline{A \vec{\zeta}}$$

$$\Rightarrow \left(\underline{A} - \lambda \underline{\underline{I}} \right) \vec{\zeta} = \vec{\eta}.$$

Fundamental Matrix (5.7)

suppose we have for the $n \times n$ system

$$\frac{d\vec{v}}{dt} = A\vec{v}$$

that

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n.$$

is a general soln, then $\vec{v}$ may be written in matrix form as

$$\vec{v} = \underbrace{\begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{pmatrix}}_{\text{called fundamental matrix } \underline{\underline{\Phi}}(t)} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}.$$

to find $\vec{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ apply IC's:

$$\vec{v}(0) = \vec{v}_0 = \underline{\underline{\Phi}}(0)\,\vec{c}.$$

$$\vec{c} = \underline{\underline{\Phi}}(0)^{-1}\vec{v}_0.$$

so we have

$$\vec{v} = \underline{\underline{\Phi}}(t)\,\underline{\underline{\Phi}}(0)\,\vec{c}.$$