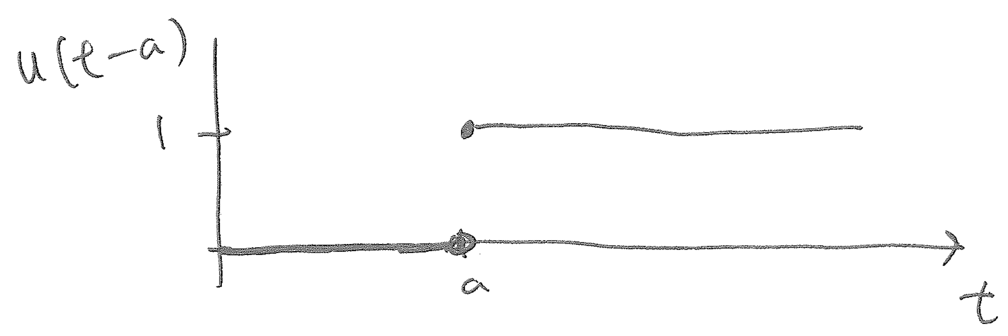
Periodic and piecewise cont's forcing functions (4.5)

previously, we have dealt mostly with continous forcing function ($f(t)$ on the RHS). However, this is not always the case, e.g. spring-mass system is suddenly kicked, or a switch is flipped on in an electric circuit. Here, the forcing fcn is 0 for some time, then becomes nonzero almost instantaneously. Simple example is the unit step fcn (Heavide step fcn):

$$u_a(t) = u(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$$



from before,

$$\mathcal{L}(u(t-a)) = \int_0^\infty e^{-st}\, u(t-a)\, dt$$

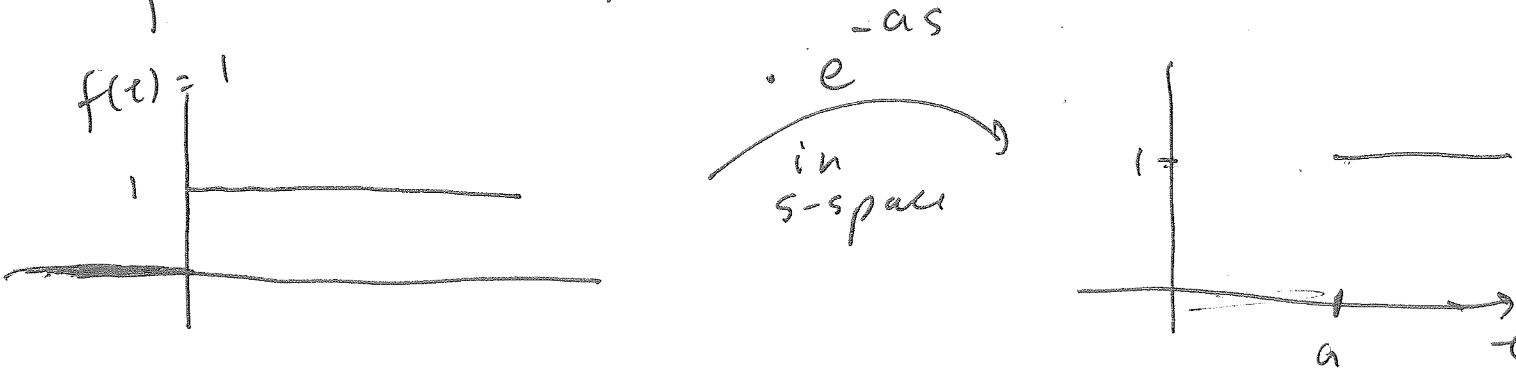
$$= \int_0^a 0\, dt + \int_a^{\cancel{t}\,\infty} e^{-st} \cdot 1\, dt$$

$$= \frac{1}{s} e^{-as}.$$

recall that $\mathcal{L}(1) = \frac{1}{s}$. so

multiplication by e^{-as} in s-space $\Rightarrow$

"shift" in t-space



[conversely, multiplication by e^{at} in t-space

$\Rightarrow$ shift in s-space $\left(\mathcal{L}(e^{at} f(t)) = F(s-a)\right)$]

Thm 1 – translation on t-axis

if $\mathcal{L}(f(t))$ exists, then

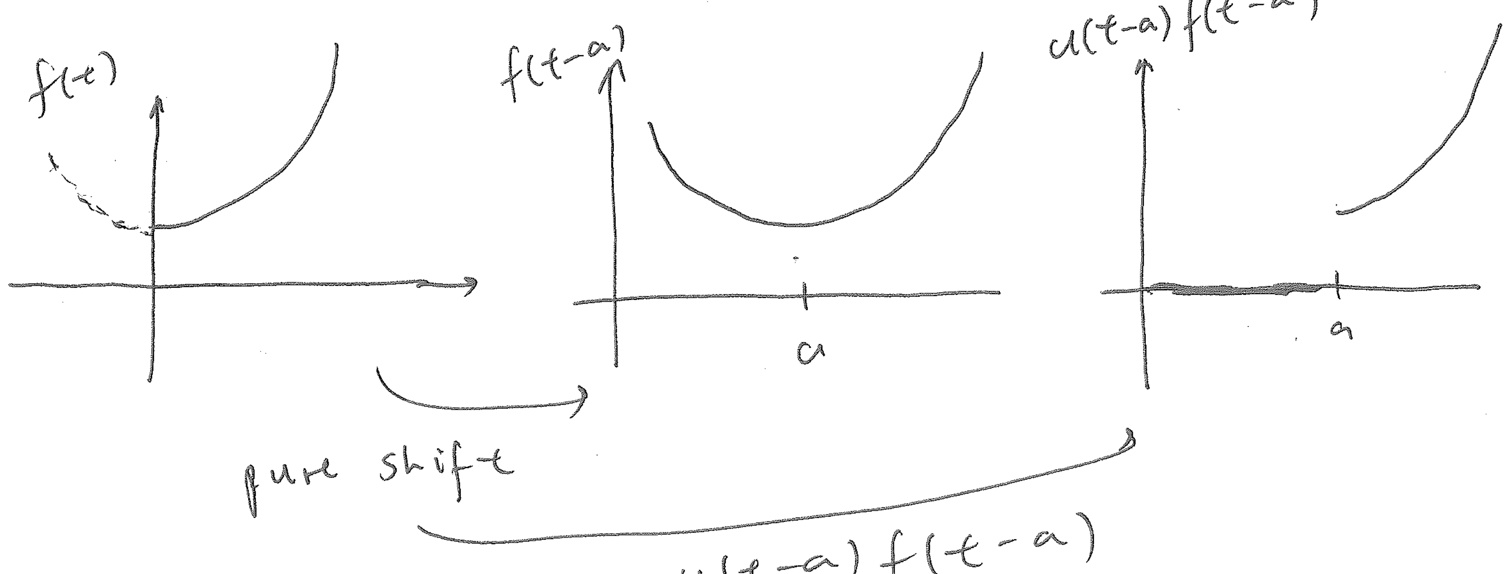
$$\mathcal{L}\left(u(t-a)\,f(t-a)\right) = e^{-as}F(s)$$

where $F(s) = \mathcal{L}(f(t))$

or

$$\mathcal{L}\left(e^{-as}F(s)\right) = u(t-a)\,f(t-a)$$

note that $u(t-a)f(t-a)$ is not a pure shift



proof: $\quad e^{-as}F(s) = e^{-as}\displaystyle\int_0^\infty e^{-st} f(t)\,dt.$

$$= \int_0^\infty e^{-(a+t)s} f(t)\, dt.$$

$\sigma = a+t$, $d\sigma = dt$ $\quad \sigma : a \to \infty$

$$e^{-as} F(s) = \int_a^\infty e^{-\sigma s} f(\sigma - a)\, d\sigma.$$

$$= \int_0^\infty e^{-\sigma s} u(\sigma - a) f(\sigma - a)\, d\sigma$$

$$= \mathcal{L}\left(u(t-a) f(t-a) \right)$$

$\square$

<u>Ex</u>

find $\mathcal{L}(f(t))$ where

$$f(t) = \begin{cases} \sin t & 0 \le t < \pi/4 \\ \sin t + \cos(t - \pi/4) & t \ge \pi/4 \end{cases}$$

$$f(t) = \sin t + \begin{cases} 0 & 0 \le t \le \pi/4 \\ \cos(t - \pi/4) & t \ge \pi/4. \end{cases}$$

$$= \sin t + \cos(t - \pi/4) \left\{ \begin{array}{ll} 0 & 0 \le t \le \pi/4 \\ 1 & t \ge \pi/4 \end{array} \right.$$

$\uparrow u(t - \pi/4)$

$$= \sin t + u(t - \pi/4)\cos(t - \pi/4)$$

$$\mathcal{L}(f) = \mathcal{L}(\sin t) + \mathcal{L}(u(t - \pi/4)\cos(t - \pi/4))$$

to compute $\mathcal{L}(u(t - \pi/4)\cos(t - \pi/4))$,

use Thm 1: calculate $\mathcal{L}(\cos t)$ then

multiply by $e^{-\frac{\pi}{4}s}$

$$\mathcal{L}(f) = \frac{1}{1+s^2} + e^{-\frac{\pi}{4}s}\frac{s}{s^2+1}$$

<u>EX</u> find the inverse transform of

$$F(s) = \frac{1 - e^{-2s}}{s^2}$$

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$

$$F(s) = \frac{1}{s^2} - e^{-2s}\frac{1}{s^2}$$

$$\mathcal{L}^{-1}(F) = \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) - \mathcal{L}^{-1}\left(e^{-2s}\frac{1}{s^2}\right).$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = t \Rightarrow \mathcal{L}^{-1}\left(e^{-ss}\frac{1}{s^2}\right) =$$

$$u(t-2)\,(t-2)$$

$$\mathcal{L}^{-1}(F) = t - u(t-2)(t-2)$$

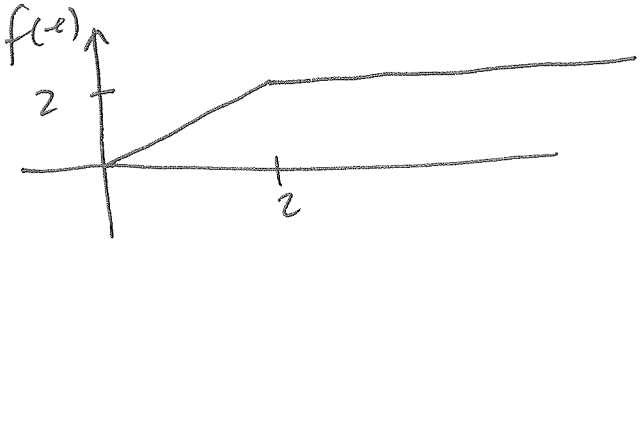
$$= \begin{cases} t - 0 & 0 \le t < 2 \\ t - (t-2) & t \geq 2 \end{cases}$$

$u(t-2)t^2$

$u(t-2)(t-2)^2$

$- \ldots$

$$= \begin{cases} t & 0 \le t < 2 \\ 2 & t \geq 2. \end{cases}$$



Ex compute $\mathcal{L}(f(t))$ where

$$f(t) = \begin{cases} t & 0 \le t \le t_0 \\ 0 & t \geq t_0 \end{cases}$$

$$f(t) = t + \begin{cases} 0 & 0 \le t \le t_0 \\ -t & t \geq t_0 \end{cases}$$

$$= t - t\,u(t - t_0) = t - u(t - t_0)\,t$$

$$= t - u(t - t_0)(t - t_0 + t_0)$$

$$= t - u(t - t_0)(t - t_0) - t_0\,u(t - t_0)$$

$$\Rightarrow \mathcal{L}(f) = \frac{1}{s^2} - e^{-t_0 s}\frac{1}{s^2} - \frac{t_0}{s}e^{-t_0 s}$$

Ex (p.306) compute $\mathcal{L}(g(t))$ where

$$g(t) = \begin{cases} 0 & 0 \le t < 3 \\ t^2 & t \geq 3 \end{cases}$$

$$g(t) = u(t-3)\,t^2 = u(t-3)\left[(t-3)^2 + 6t - 9\right]$$

$$= u(t-3)\left[(t-3)^2 + 6(t-3) + 9\right]$$

$$= u(t-3)(t-3)^2 + 6u(t-3)(t-3) + 9u(t-3)$$

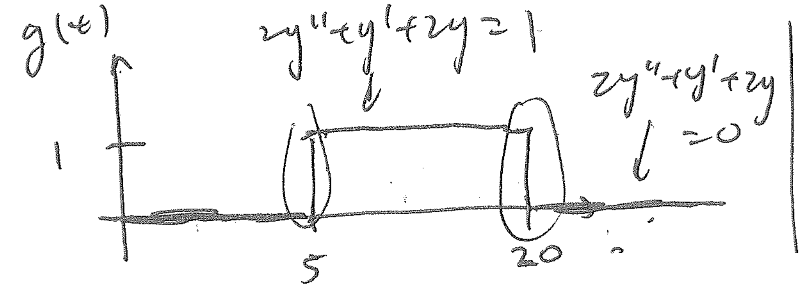
$$\mathcal{L}(g) = \frac{2}{s^3}e^{-3s} + 6\frac{1}{s^2}e^{-3s} + \frac{9}{s}e^{-3s}$$

$$\left(\mathcal{L}(t^n) = \frac{n!}{s^{1+n}}\right)$$

Laplace transform allows one to solve ODE with piecewise defined forcing function in one step (as opposed solving in different intervals and then stitching together)

<u>Ex</u> solve $2y'' + y' + 2y = g(t)$, $y(0) = y'(0) = 0$

where $g(t) = u(t-5) - u(t-20)$



immediately can infer that $y=0$ for $t \in [0,5]$.

apply $\mathcal{L}(\cdot)$:

$$Y(s)\left[2s^2+s+2\right] = \frac{e^{-5s}}{s} - \frac{e^{-20s}}{s}$$

$$Y(s) = \frac{e^{-5s}-e^{-20s}}{s\left[2s^2+s+2\right]}$$

$$= e^{-5s}H(s) - e^{-20s}H(s).$$

where $H(s) = \dfrac{1}{s\left[2s^2+s+2\right]}$

approach: find $h(t) = \mathcal{L}^{-1}(H(s))$ then apply Thm 1 to invert. That is,

$$y(t) = u(t-5)\,h(t-5) - u(t-20)\,h(t-20)$$

$$H(s) = \frac{1}{s(2s^2+s+2)} = \frac{A}{s} + \frac{Bs+C}{2s^2+s+2}$$

$$A = \frac{1}{2}, \quad B = -1, \quad C = -\frac{1}{2}$$

then

$$H(s) = \frac{1/2}{s} - \frac{s + 1/2}{2s^2 + s + 2}$$

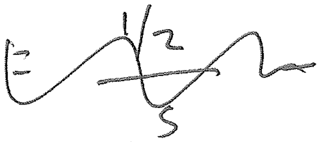
$$= \frac{1/2}{s} - \frac{s + 1/2}{2\left[s^2 + \frac{1}{2}s + 1 \right]}$$

$$= \frac{1/2}{s} - \frac{s + 1/2}{2\left[s^2 + \frac{1}{2}s + \frac{1}{16} - \frac{1}{16} + 1 \right]}$$

$$= \frac{1/2}{s} - \frac{s + 1/2}{2\left[(s + \frac{1}{4})^2 + \frac{15}{16} \right]}$$

$$= \frac{1/2}{s} - \frac{s + 1/4 + 1/4}{2\left[(s + 1/4)^2 + \frac{15}{16} \right]}$$

$$= \frac{1/2}{s} - \frac{1}{2} \frac{s + 1/4}{(s + \frac{1}{4})^2 + \frac{15}{16}} - \frac{1}{8}\sqrt{\frac{16}{15}} \frac{\sqrt{15/16}}{(s + \frac{1}{4})^2 + \frac{15}{16}}$$



therefore

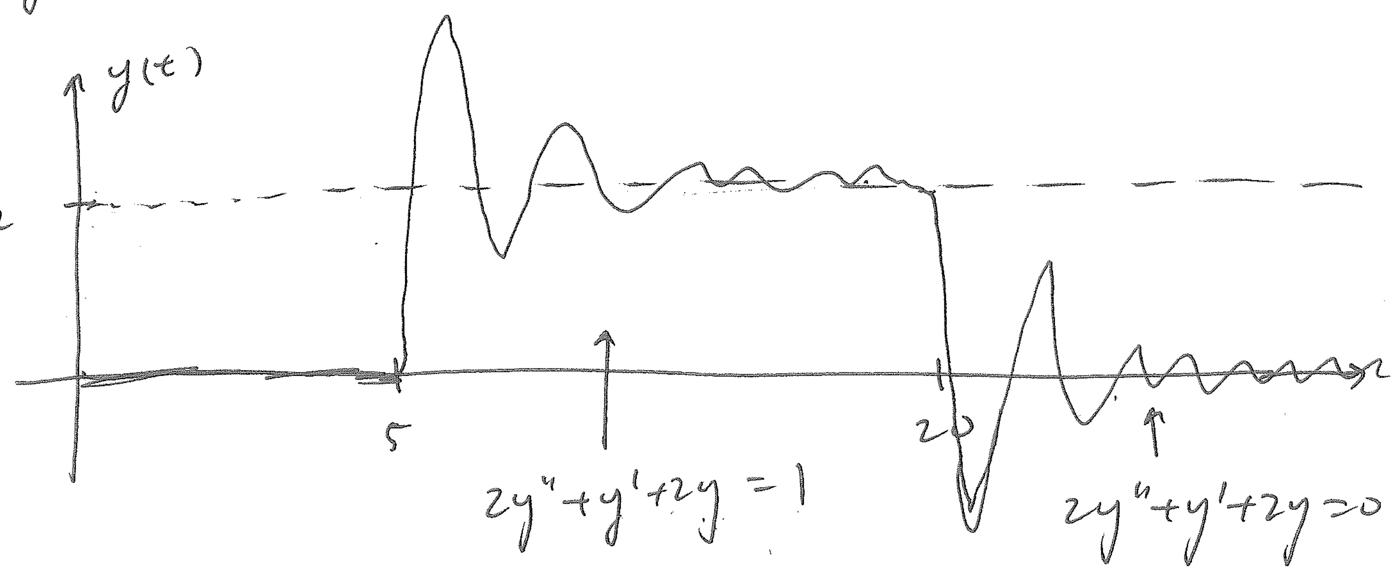
$$h(t) = \frac{1}{2} - \frac{1}{2} e^{-t/4} \cos\sqrt{\frac{15}{16}}\, t - \frac{1}{8}\cdot\frac{4}{\sqrt{15}} e^{-t/4} \sin\sqrt{\frac{15}{16}}\, t$$

$$h(t) = \frac{1}{2}\left[1 - e^{-t/4}\cos\sqrt{\frac{15}{16}}\, t - \frac{1}{\sqrt{15}} e^{-t/4} \sin\sqrt{\frac{15}{16}}\, t\right]$$

$t \to t-5$ $\quad$ $t \to t-20$

then

$$y(t) = u(t-5)\,h(t-5) - u(t-20)\,h(t-20)$$



decaying oscillations about $y = 1/2$

decay osc. about $y = 0$.

Thm 2 - transform of periodic fcns

let $f(t)$ be periodic with period $p > 0$ ($f(t \pm np) = f(t)$ for integer n) and piecewise cont's for $t \geq 0$. Then transform $F(s)$ is given by

$$F(s) = \frac{1}{1 - e^{-ps}} \int_0^p e^{-st} f(t)\, dt.$$

proof:

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \left[\int_0^p + \int_p^{2p} + \int_{2p}^{3p} + \cdots \right] e^{-st} f(t)\, dt.$$

$$= \sum_{n=0}^{\infty} \int_{np}^{(n+1)p} e^{-st} f(t)\, dt.$$

let $I_n \equiv \int_{np}^{(n+1)p} e^{-st} f(t)\, dt.$

let ~~then~~ $\tau = t - np \Rightarrow d\tau = dt$

$\tau : 0 \to p.$

$$I_n = \int_0^p e^{-s(\tau+np)} \underbrace{f(\tau+np)}_{f(\tau) \text{ by periodicity}} d\tau$$

$$= e^{-snp} \int_0^p e^{-s\tau} f(\tau)\, d\tau$$

$$\Rightarrow F(s) = \sum_{n=0}^{\infty} e^{-snp} \underbrace{\boxed{\int_0^p e^{-st} f(t)\, dt}}_{\text{indep. of } n}$$

$$= \int_0^p e^{-st} f(t)\, dt \underbrace{\left[\sum_{n=0}^{\infty} (e^{-sp})^n \right]}_{\frac{1}{1-e^{-sp}}}$$

now recall,

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \quad \text{for } |z| < 1$$

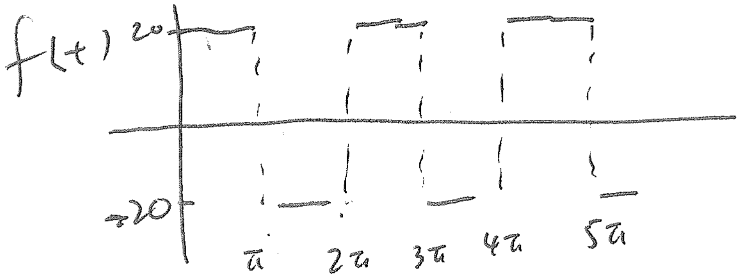
$$\Rightarrow F(s) = \frac{1}{1-e^{-sp}} \int_0^p e^{-st} f(t)\, dt.$$

$\square$

EX (p. 311)

solve $x'' + 4x' + 20x = f(t)$; $x(0) = x'(0) = 0$

where $f(t)$ is the square wave function of amplitude 20 and period 2π



apply $\mathcal{L}(\cdot)$: $\mathcal{L}(f(t)) = \dfrac{1}{1-e^{-2\pi s}} \displaystyle\int_0^{2\pi} e^{-st} f(t)\,dt$

$$= \frac{1}{1-e^{-2\pi s}}\left[\int_0^{\pi} e^{-st}\cdot 20\,dt + \int_{\pi}^{2\pi} e^{-st}(-20)\,dt\right]$$

$$= \frac{1}{1-e^{-2\pi s}}\left[20\left(-\frac{1}{s}e^{-st}\right)\Big|_0^{\pi} - 20\left(\frac{-1}{s}e^{-st}\right)\Big|_{\pi}^{2\pi}\right]$$

$$= \frac{20}{1-e^{-2\pi s}}\left[\frac{1-e^{-\pi s} + e^{-2\pi s} - e^{-\pi s}}{s}\right].$$

$$= \frac{20}{s(1-e^{-2\pi s})}\left[1 - 2e^{-\pi s} + e^{-2\pi s}\right]$$

let $y = e^{-\pi s}$

then

$$F(s) = \frac{20}{s}\,\frac{(1-2y+y^2)}{1-y^2}$$

$$= \frac{20}{s}\,\frac{(1-y)^2}{(1-y)(1+y)} = \frac{20}{s}\,\frac{(1-y)}{1+y}$$

$$= \frac{20}{s}\left(\frac{1-e^{-\pi s}}{1+e^{-\pi s}}\right)$$

$$\cancel{= \frac{20}{s}(1-e^{-\pi s})\left(1-e^{-2\pi s}+e^{-4\pi s}-e^{-6\pi s}+\cdots\right)}$$

$$= \frac{20}{s}(1-e^{-\pi s})\left(1-e^{-\pi s}+e^{-2\pi s}-e^{-3\pi s}+\cdots\right)$$

$$= \frac{20}{s}\Big[1-e^{-\pi s}+e^{-2\pi s}-e^{-3\pi s}+\cdots$$

$$-e^{-\pi s}+e^{-2\pi s}-e^{-3\pi s}+\cdots\Big]$$

$$= \frac{20}{s}\Big[1 - 2e^{-\pi s} + 2e^{-2\pi s} - 2e^{-3\pi s} + 2e^{-4\pi s} - \cdots\Big]$$

$$= \frac{20}{s} + \frac{40}{s}\sum_{n=1}^{\infty}(-1)^n e^{-n\pi s}.$$

so from ODE, we have (0 IC's)

$$(s^2+4s+20)X(s) = F(s).$$

$$s^2+4s+4+16 = (s+2)^2+16$$

so

$$X(s) = G(s) + 2\sum_{n=0}^{\infty}(-1)^n e^{-n\pi s}G(s)$$

where

$$G(s) = \frac{20}{s\left[(s+2)^2+16\right]}$$

$$= \frac{1}{s} - \frac{s+4}{(s+2)^2+16}$$

$$= \frac{1}{s} - \frac{s+2}{(s+2)^2+16} + \frac{1}{2}\frac{4}{\cdots}$$

$$g(t) = 1 - e^{-2t}\cos 4t - \frac{1}{2}e^{-2t}\sin 4t.$$

then

$$x(t) = g(t) + 2\sum_{n=1}^{\infty}(-1)^n u(t-n\pi)\,g(t-n\pi)$$