Method of undetermined coefficients

(2.5) - for nonhomogeneous linear ODE's with constant coeff's

- up until now, we have been learning how to solve <u>homogeneous</u> equations (ie $L(y) = \underline{\underline{0}}$)

- <u>nonhomogeneous</u> equations arise in examples such as externally forced spring or pendulum, or non-constant voltage source in an RLC circuit (see p. M1-M5)

- such equations have the form

$$L(y) = \underline{\underline{f(t)}}$$

- by linearity of the operator L $\left[L(c_1 y_1 + c_2 y_2) = c_1 L(y_1) + c_2 L(y_2) \right]$ the solution y can be written as

$$y = y_h + y_p$$

y_p: particular soln (no arbitrary constants)

y_h: homogeneous solution (book: complementary) (contains arbitrary constants) where

$$\left.\begin{aligned} L(y_h) &= 0 \\ L(y_p) &= f(t) \end{aligned}\right\} \quad \begin{aligned} L(y) &= L(y_h + y_p) \\ &= L(y_h) + L(y_p) \\ &= 0 + f(t) \\ &= f(t) \end{aligned}$$

- [we know from previous how to obtain y_h]

- furthermore, if $f(\overset{t}{\cancel{x}}) = f_1(t) + f_2(t)$,
  then y_p may be written as

  $y_p = y_{p_1} + y_{p_2}$, where

$$L(y_{p_1}) = f_1(t)$$
$$L(y_{p_2}) = f_2(t)$$

- we will cover ~~two~~ two methods for obtaining y_p

- the first is method of undetermined coeff's (MUC)

- involves making an educated guess for the form of y_p

- need constant coeff's in ODE

- <u>only</u> <u>works</u> when $f(t)$ is a finite linear combination of polynomials, exponentials, and sines and cosines

e.g. $f(t) = (t^3 + 2t^2 + 1)e^{5t}\cos t + x^7 e^{-3t} \sin 2t$

- the reason is derivatives of polynomials are polynomials, and derivatives of exp's (and by extension sines, cosines) are exp's.

- involves solving algebraic problem "easy"

- second method is called variation of parameters (VOP)
  - works for any $f(t)$
  - " " non-constant coeff's
  - no guessing required
  - requires integration ("difficult")
- we do MUC first

Ex
$$y'' - 3y' - 4y = 3e^{2t}$$

find the general solution

first find homogeneous solution

C.E. $r^2 - 3r - 4 = 0$
no r

$\Rightarrow r = 4, -1 \Rightarrow y_1 = e^{4t}, \; y_2 = e^{-t}$

$y_h = c_1 e^{4t} + c_2 e^{-t}$

note: $f(t) = 3e^{2t}$ is not a multiple of either y_1 or y_2 *

so we guess: $y_p(t) = a\,e^{2t}$ for some constant to be found. Since $L(y_p) = f(t)$, sub y_p into DE to find a: $y_p' = 2ae^{2t}$, $y_p'' = 4ae^{2t}$.

$$4ae^{2t} - 3(2ae^{2t}) - 4ae^{2t} = 3e^{2t}$$

equate of e^{2t}:

$$4a - 6a - 4a = 3$$

$$\Rightarrow a = -\frac{1}{2}$$

$$\Rightarrow y_p = -\frac{1}{2}e^{2t}$$

general solution is then

$$y = \underbrace{c_1 e^{4t} + c_2 e^{-t}}_{y_h} \underbrace{- \frac{1}{2}e^{2t}}_{y_p}$$

notes:

1) still only 2 arbitrary constants

2) c_1 and c_2 from IC's (always do last)

3) if $f(t) = 5e^{4t}$ or $-7e^{-t}$ (some multiple of y_1 or y_2)

cannot guess $y_p = ae^{4t}$ or $y_p = ae^{-t}$ since $L(e^{4t}) = L(e^{-t}) = 0$.

(discuss later).

4) if $f(t) = e^{r_p t}$, where $e^{r_p t}$ is <u>not</u> a homogeneous solution,

guess $y_p = a e^{r_p t}$ for some a to be found

<u>EX</u> find particular solution of

$$y'' - 3y' - 4y = 2\sin t.$$

still ~~need~~ need to find y_1 and y_2 first to make sure $f(t)$ is not a homog. soln. ($y_1 = e^{4t}$, $y_2 = e^{-t}$, so we are okay)

naive guess: $y_p = a\sin t$, $y_p' = a\cos t$

$y_p'' = -a\sin t$.

sub into ODE:

$$-5a\sin t - 3a\cos t = 2\sin t.$$

$$\Rightarrow \quad \left.\begin{array}{l} -5a = 2 \\ -3a = 0 \end{array}\right\} \text{ inconsistent.}$$

we made the wrong guess. due to the $-3y'$ term in the ODE, we must (in general) include both sine and cosine in guess so $y_p = a_1 \cos t + a_2 \sin t.$

$y_p' = -a_1 \sin t + a_2 \cos t.$

$y_p'' = -a_1 \cos t - a_2 \sin t.$

sub into ODE:

$$\cos t\left[-a_1 - 3a_2 - 4a_1\right] + \sin t\left[-a_2 + 3a_1 - 4a_2\right] = 2\sin t$$

equate coefficients:

cos t ~~sint~~: $-5a_1 - 3a_2 = 0$

sin t ~~cost~~: $3a_1 - 5a_2 = 2$

$$a_2 = \cancel{\tfrac{3}{5}a_1} - \frac{5a_1}{3}$$

$$3a_1 - 5\left[-\frac{5a_1}{3}\right] = 2$$

$$3a_1 + \frac{25}{3}a_1 = 2 \Rightarrow a_1 = \frac{3}{17}$$

$$\frac{34a_1}{3} = 2$$

$$a_1 = \frac{6}{34} = 3/17$$

check this.

$$\rightarrow \quad a_2 = -\frac{5}{3}a_1 = -\frac{5}{3}\left(\frac{3}{17}\right) = \frac{-5}{17}.$$

$$y_p = \frac{3}{17}\cos t - \frac{5}{17}\sin t.$$

general soln: $y = c_1 y_1 + c_2 y_2 + y_p.$

apply IC's here...

Ex find particular soln of.

$$y'' - 3y' - 4y = -8\boxed{e^t \cos 2t.}$$

[we have $y_1 = e^{4t}$, $y_2 = e^{-t}$]

$$\cancel{y_p = a_1 e^t + a_2 \sin 2t. + a_3 \cos 2t}.$$

$$-8e^t \cos 2t = -8e^t \left[\frac{e^{2it} + e^{-i2t}}{2}\right].$$

$$= -8\left[\frac{e^{(1+2i)t} + e^{(1-2i)t}}{2}\right].$$

$$= -4e^{(1+2i)t} - 4e^{(1-2i)t}.$$

$$y_p = \tilde{a}_1 e^{(1+2i)t} + \tilde{a}_2 e^{(1-2i)t}$$

$$= \cancel{\tilde{a}_1 e^{t}}\, e^t\left[\tilde{a}_1 e^{i2t} + \tilde{a}_2 e^{-i2t}\right].$$

$$= e^t\left[\underbrace{(\tilde{a}_1 + \tilde{a}_2)}_{a_1}\cos 2t + \underbrace{i(\tilde{a}_1 - \tilde{a}_2)}_{a_2}\sin t\right].$$

$$= e^t\left[\cancel{a_1}\; a_1\cos 2t + a_2 \sin 2t\right].$$

so this is the form of the guess.

put into ODE. (algebra tedious)

$$y_p' = (a_1 + 2a_2)e^t\cos 2t + (-2a_1 + a_2)e^t\sin 2t.$$

$$y_p'' = (-3a_1 + 4a_2)e^t\cos 2t + (-4a_1 - 3a_2)e^t\sin 2t$$

$\cancel{10a_1 + 2a_2}$

equate coeff's of $e^t\cos 2t$ and $e^t\sin 2t$.

$$\Rightarrow \left.\begin{array}{l} 10a_1 + 2a_2 = 8 \\ 2a_1 - 10a_2 = 0 \end{array}\right\} \Rightarrow \begin{array}{l} a_1 = \frac{10}{13} \\ a_2 = \frac{2}{13}. \end{array}$$

$$y_p = \frac{10}{13} e^t \cos 2t + \frac{2}{13} e^t \sin 2t.$$

$$y = c_1 y_1 + c_2 y_2 + y_p.$$

apply IC's ...

EX

$$y'' - 3y' - 4y = 4t^2 \underbrace{e^0}_{1}$$

notice, e^0 is not a homog. soln.

guess: $y_p = \underbrace{a_0 + a_1 t + a_2 t^2}_{\text{must include all lower powers.}}$.

$$y_p' = a_1 + 2a_2 t$$

$$y_p'' = 2a_2$$

$$2a_2 - 3(a_1 + 2a_2 t) - 4(a_0 + a_1 t + a_2 t^2) = 4t^2$$

equate coeff's of t^0, t^1, t^2:

$$t^2 : \quad -4a_2 = 4 \quad \Rightarrow \quad a_2 = -1.$$

$$t : \quad -4a_1 - 6a_2 = 0 \quad \Rightarrow \quad a_1 = \frac{3}{2}.$$

$$t^0 : \quad 2a_2 - 3a_1 - 4a_0 = 0$$

$$\Rightarrow \quad a_0 = -\frac{13}{8}$$

then $y = c_1 e^{4t} + c_2 e^{-t} + \left[\frac{-13}{8} + \frac{3}{2}t - t^2 \right].$

<u>EX</u> find general soln of

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin t - 8e^t \cos 2t + 4t^2.$$

by linearity: we may write soln as.

$y = c_1 y_1 + c_2 y_2 + y_{p_1} + y_{p_2} + y_{p_3} + y_{p_4}$. where

$L(y_{p_1}) = 3e^{2t}, \quad L(y_{p_2}) = 2\sin t.$

$L(y_{p_3}) = -8e^t \cos 2t, \quad L(y_{p_4}) = 4t^2.$

$$\Rightarrow y = c_1 y_1 + c_2 y_2 - \frac{1}{2} e^{2t} + \frac{3}{17}\cos t$$

$$- \frac{5}{17}\sin t + \frac{10}{13} e^{t}\cos 2t$$

$$+ \frac{2}{13} e^{t}\sin 2t - \frac{13}{8} + \frac{3}{2}t$$

$$-t^2.$$

EX

find general solution of.

$$y'' + 4y = 3\cos 2t.$$

first find y_1, y_2:

$$y_h'' + 4y_h = 0.$$

C.E: $r^2 + 4 = 0 \Rightarrow r = \pm 2i.$

$$y_1 = \cos 2t, \quad y_2 = \sin 2t.$$

for y_p, can we guess

$$y_p = a_1 \cos 2t + a_2 \sin 2t \ ?$$

<u>no</u> because these are homog.

soln's.

so, we augment our guess by 1 power of t:

$$y_p = t\,[a_1 \cos 2t + a_2 \sin 2t].$$

notice that y_p' and y_p'' contain terms involving $\cos 2t$ and $\sin 2t$, which match the $3\cos 2t$ term on the rhs.

sub this form ~~in~~ for y_p into ODE.

$$y_p' = a_1 \cos 2t + a_2 \sin 2t + t\,[-2a_1 \sin 2t + 2a_2 \cos 2t].$$

$$y_p'' = t\,[-4a_1 \cos 2t - 4a_2 \sin 2t] + 2\,[-2a_1 \sin 2t + 2a_2 \cos 2t].$$

now with $y_p'' + 4y_p = 3\cos 2t$.

$$t\left[-4a_1 \cancel{\cos 2t} - 4a_2 \cancel{\sin 2t}\right]$$

$$+2\left[-2a_1 \sin 2t + 2a_2 \cos 2t\right]$$

$$+4t\left[\cancel{a_1 \cos 2t} + \cancel{a_2 \sin 2t}\right] = 3\cos 2t.$$

the $t\cos 2t$ and $t\sin 2t$ terms <u>must</u> <u>cancel</u> ... why?

now equate coeff's :

$$\left.\begin{aligned} 4a_2 &= 3 \\ -4a_1 &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} a_2 &= \tfrac{3}{4} \\ a_1 &= 0. \end{aligned}$$

then we have

$$y = \underbrace{c_1 \cos 2t + c_2 \sin 2t}_{y_h} + \underbrace{\tfrac{3}{4} t \sin 2t}_{y_p}.$$

IC's . . . .

summary

for nonhomog. linear equation with constant coeff's,

$$ay'' + by' + cy = f(t), \quad y(t_0) = y_0, \quad y'(t_0) = v_0.$$

where

$$f(t) = P_n(t)\, e^{\alpha t} \sin \beta t$$

or

$$f(t) = P_n(t)\, e^{\alpha t} \cos \beta t,$$

where $P_n(t)$ is an n-th order polynomial

do the following:

(1) solve homog. eqn to obtain y_1, y_2.

(2)* make guess

$$y_p(t) = t^s e^{\alpha t} \left[(a_0 + a_1 t + \dots + a_n t^n) \cos \beta t + (b_0 + b_1 t + \dots + b_n t^n) \sin \beta t \right]$$

(3)' (α, β could be 0)

(3) sub y_p into ODE and solve for $a_0, \ldots, a_n$, $b_0, \ldots, b_n$ by equating coeff's of $t^k e^{\alpha t} \sin \beta t$, $k = 0, \ldots, n$
$t^k e^{\alpha t} \cos \beta t$

(4) write general soln
$$y = c_1 y_1 + c_2 y_2 + y_p.$$

(5) apply IC's to obtain c_1, c_2.

*(2) if $\beta = 0$, $f(t) = P_n(t) e^{\alpha t}$.
$$y_p = t^s e^{\alpha t} \left[a_0 + a_1 t + \ldots a_n t^n \right].$$

if $\alpha = \beta = 0$, $f(t) = P_n(t)$,
$$y_p(t) = t^s (a_0 + \ldots + a_n t^n).$$

s is the smallest integer s.t. no term in $y_p(t)$ is a multiple of y_1, y_2

for second order, $s = 0, 1, 2$.