(VI)

Variation of parameters (2-5)

(vop).

- a general method for obtaining particular solution of a linear (second order for us) equation with possibly non-constant coeff's.

$$L(y) = y'' + p(t)y' + q(t)y = f(t)$$

- requires we know y_1, y_2 (homog. soln's)
- " evaluation of integral

Idea: recall, if $f(t) \equiv 0$, then

$$y = c_1 y_1 + c_2 y_2 \qquad c_{1,2} \text{ constants}$$

with vop, let particular solution y_p be written

$$y_p = u_1(t)\, y_1 + u_2(t)\, y_2$$

and find u_1, u_2 s.t.

$L(y_p) = f(t)$ ← this is one condition on u_1, u_2.

since we have 2 unknowns, we may specify a second condition that leads to a first order equation for u_1 and u_2 (ie., eliminate u_1'' and u_2'' terms)

so

$$y_p' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2'$$

$$= u_1 y_1' + u_2 y_2' + \underbrace{u_1' y_1 + u_2' y_2}_{\text{set this to } 0}$$

let $\boxed{u_1' y_1 + u_2' y_2 = 0}$ 1 condition

V3

$$\Rightarrow \quad y_p' = u_1 y_1' + u_2 y_2'$$

$$y_p'' = u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2''$$

sub into ODE:

$$u_1 y_1'' \boxed{+ u_2 y_2''} + u_1' y_1' + u_2' y_2'$$

$$+ p(t)\left[u_1 y_1' \boxed{+ u_2 y_2'} \right] + q(t)\left[u_1 y_1 \boxed{+ u_2 y_2} \right]$$

$$= f(t)$$

$$\cancel{u_1 \left[y_1'' + p(t) y_1' + q(t) y_1 \right]}$$

$$\cancel{+ u_2 \left[y_2'' + p(t) y_2' + q(t) y_2 \right]}$$

y_1, y_2 are homog. soln's $\rightarrow L(y_{1,2}) = 0$

$$+ \quad u_1' y_1' + u_2' y_2' = f(t)$$

$$\Rightarrow \quad \boxed{u_1' y_1' + u_2' y_2' = f(t)}$$ 2nd condition

V4

2 eqn/s, 2 unknowns (u_1', u_2')

$$\begin{cases} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = f(t) \end{cases}$$

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ f(t) \end{pmatrix}.$$

Cramer's rule:

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_2' & y_2' \end{vmatrix}} = \frac{-y_2 f}{W}.$$

wronskian

$W(y_1, y_2; t)$

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f \end{vmatrix}}{W} = \frac{y_1 f}{W}.$$

$$\boxed{\begin{aligned} u_1(t) &= -\int \frac{y_2(t)\, f(t)}{W(y_1, y_2; t)}\, dt. \\ u_2(t) &= \int \frac{y_1(t)\, f(t)}{W(y_1, y_2; t)}\, dt. \end{aligned}}$$

can take constants of integration to be 0 wlog.

* require y_1, y_2

<u>Ex</u> find general solution of

$$y'' + y = \tan t \qquad \left(-\frac{\pi}{2} < t < \frac{\pi}{2}\right)$$

first find y_1, y_2 :

$$y_1 = \cos t, \quad y_2 = \sin t.$$

V6

let $y_p = u_1(t)\, y_1 + u_2(t)\, y_2.$

calculate $W(y_1, y_2; t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$$= \begin{vmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{vmatrix} = -\sin^2 t - \cos^2 t$$

$$= -1.$$

$$u_1(t) = -\int \frac{y_2(t) f(t)}{W}\, dt$$

$$= -\int \frac{\cos t \tan t}{-1}\, dt = \int \sin t\, dt$$

$$= -\cos t + 0$$

$\uparrow$ wlog.

$$u_2(t) = \int \frac{y_1 f(t)}{-1}\, dt = -\int \sin t \tan t\, dt$$

$$= \int -\frac{\sin^2 t}{\cos t}\, dt = -\int \frac{1 - \cos^2 t}{\cos t}.$$

$$= -\int \sec t - \cos t \, dt.$$

$$\vdots \text{ etc}$$

$$= \sin t - \log(\sec t + \tan t) + \underset{\uparrow \atop \text{wlog.}}{0}$$

then

$$y_p = -\cos t \sin t + \left[\sin t - \log(\sec t + \tan t)\right] \cdot \cos t$$

$$= -\cos t \log(\sec t + \tan t).$$

$$y = c_1 y_1 + c_2 y_2 + y_p.$$