Forced oscillations and resonance (2.6) (F1)

focus on nonhomog. equations where forcing function is sinusoidal ($f(t) = F_0 \cos\omega t$, $F_0 \sin\omega t$)

undamped forced oscillations

$$m x'' + kx = F_0 \cos\omega t$$

find general solution (assume $\omega \neq \underbrace{\sqrt{\tfrac{k}{m}}}_{\text{natural frequency}} = \omega_0$)

find the homog. soln first.

$$x_h'' + \frac{k}{m} x_h = 0.$$

$$\text{c.e.: } r^2 + \frac{k}{m} = 0 \Rightarrow r = \pm i \sqrt{\frac{k}{m}} = \pm i\omega_0$$

then

$$x_h = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t.$$

$$x_p = ?$$

$$x_p = a\cos\omega t + \cancel{b\sin\omega t}.$$

since no x' term

Sub into ODE.

$x_p'' = -\omega^2 a \cos\omega t.$

$-m\omega^2 a \cos\omega t + k a \cos\omega t = F_0 \cos\omega t.$

$-m\omega^2 a + k a = F_0.$

$$a = \frac{F_0/m}{\omega_0^2 - \omega^2} \quad ; \; \omega_0^2 = \frac{k}{m}.$$

if $\omega = \omega_0$, this is not valid

$\rightarrow$ resonance ...

$$\Rightarrow \quad x(t) = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t + \frac{F_0/m}{\omega_0^2 - \omega^2}\cos\omega t$$

$$= C_1 \cos(\omega_0 t - \alpha) + \frac{F_0/m}{\omega_0^2 - \omega^2}\cos\omega t.$$

two different frequencies

what is the period?

period is least common multiple of

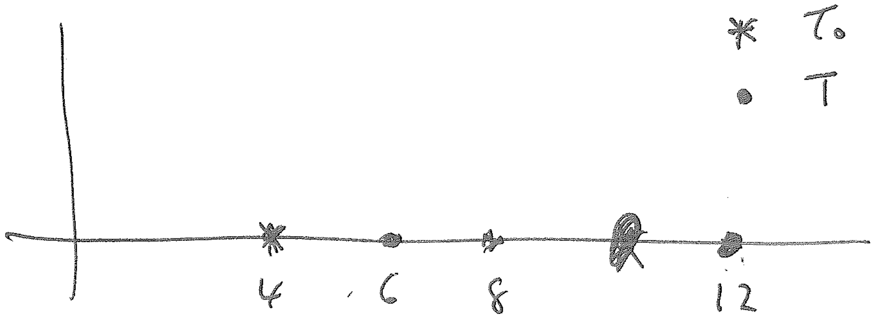
$T_0 = \frac{2\pi}{\omega_0}$, and $T = \frac{2\pi}{\omega}$

if $\omega_0 = \frac{\pi}{2}$, $\omega = \frac{\pi}{3}$,

then $T_0 = \frac{2\pi}{\pi/2} = 4$

$T = \frac{2\pi}{\pi/3} = 6$

then $T_x = 12$



Beats

suppose $x(0) = x'(0) = 0$.

then

$$x(0) = c_1 + \frac{F_0/m}{\omega_0^2 - \omega^2} = 0$$

$$c_1 = \frac{-F_0}{m(\omega_0^2 - \omega^2)}.$$

$$x'(0) = 0 \Rightarrow c_2 = 0.$$

write in more suggestive form

$$x(t) = \frac{F_0/m}{\omega_0^2 - \omega^2}\left[\cos\omega t - \cos\omega_0 t\right].$$

use trig id:

$$\cos(A-B) = \cos A\cos B + \sin A \sin B$$

$$-\quad \cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A-B) - \cos(A+B) = 2\sin A \sin B.$$

now associate:

$$\left.\begin{aligned} A - B &= \omega t. \\ A + B &= \omega_0 t \end{aligned}\right\} \Rightarrow \quad \begin{aligned} A &= \frac{\omega_0+\omega}{2} t. \\ B &= \frac{\omega_0-\omega}{2} t. \end{aligned}$$

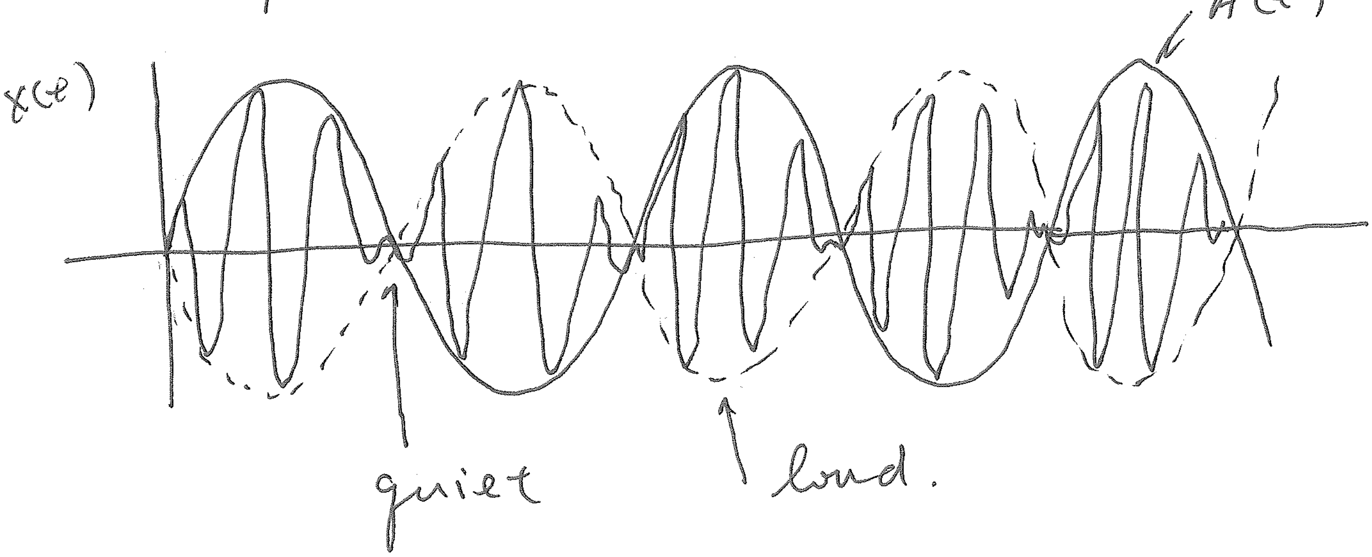
therefore

$$\cos\omega t - \cos\omega_0 t = 2\sin\frac{\omega_0+\omega}{2} t \sin\frac{\omega_0-\omega}{2} t.$$

$$x(t) = \boxed{2\,\frac{F_0/m}{\omega_0^2-\omega^2}\sin\frac{\omega_0-\omega}{2}t}\,\sin\frac{\omega_0+\omega}{2}t.$$

↑ $A(t)$: low frequency envelope

if $w_0 \approx w$, then $\frac{w_0 - w}{2}$ is very small, $\rightarrow A(t)$ is a low freq. envelope, $\frac{w_0 + w}{2}$ is large by comparison $\rightarrow$ rapid oscillations inside envelope



Resonance

notice that as $w \rightarrow w_0$ (forcing freq. very near natural freq.), $A(t)$ becomes very large. When $w = w_0$. we have resonance. Must alter MUC solution

$$x'' + \omega_0^2 x = \frac{F_0}{m} \cos \omega_0 t \quad ; \quad \omega_0^2 = \frac{k}{m}$$

homog. first : $\Rightarrow$ $x_1 = \cos \omega_0 t$

$x_2 = \sin \omega_0 t$

$$x_p(t) = t \left[A \cos \omega_0 t + B \sin \omega_0 t \right]$$

sub into ODE :

$$x_p'' = t \left[A \cos \omega_0 t + B \sin \omega_0 t \right] (-\omega_0^2)$$

$$+ 2\omega_0 \left[-A \sin \omega_0 t + B \cos \omega_0 t \right]$$

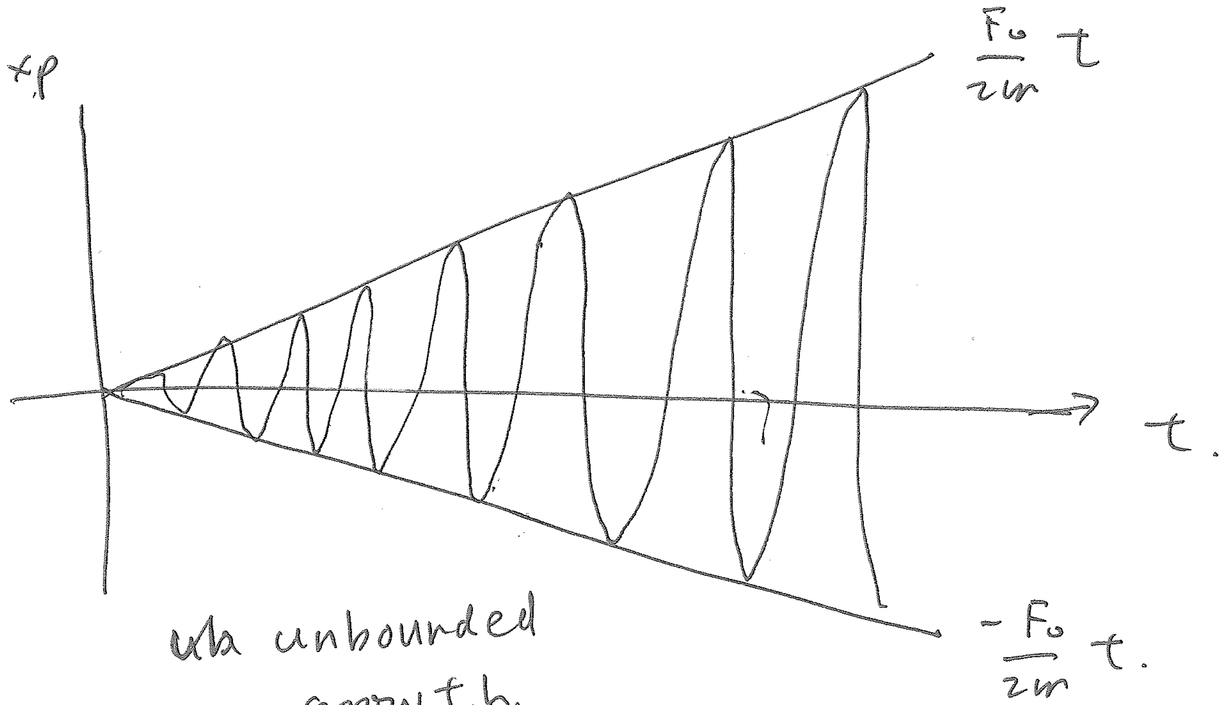
$$2\omega_0 \left[-A \sin \omega_0 t + B \cos \omega_0 t \right] = \frac{F_0}{m} \cos \omega_0 t$$

$$\Rightarrow \quad A = 0 \quad , \quad B = \frac{F_0}{2\omega_0 m}$$

$$\Rightarrow \quad x_p(t) = \frac{F_0}{2m\omega_0} t \sin \omega_0 t$$

$$x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \boxed{\frac{F_0}{2m} t \sin \omega_0 t}$$

(F7)



~~uh~~ unbounded growth.

Damped forced oscillations and practical resonane

$$mx'' + cx' + kx = F_0 \cos \omega t.$$

homog. first: $\quad$ important $c > 0$

C.E. $\quad mr^2 + cr + k = 0$

$$r = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}$$

now

$$x = x_h + x_p.$$

notice:

$x_h \sim e^{-\frac{c}{2m}t}$ so it decays in time; it is the transient solution

$\Rightarrow$ $x \to x_p$ as $t \to \infty$ since $x_h \to 0$ exp. as $t \to \infty$.

to find x_p, use MUC:

$$x_p = A\cos\omega t + B\sin\omega t.$$

$$\vdots$$

$$A = \frac{(k-m\omega^2)F_0}{(k-m\omega^2)^2+(c\omega)^2}, \quad B = \frac{c\omega F_0}{(k-m\omega^2)^2+(c\omega)^2}$$

in form

$$x_p = G\cos(\omega t - \alpha)$$

we have

$$G = \sqrt{A^2+B^2} = \frac{F_0}{\sqrt{(k-m\omega^2)^2+(c\omega)^2}}.$$

(F9).

notes :

1) G is always finite when $c>0$

2) as $\omega \to \infty$, $G(\omega) \to 0$

3) as $\omega \to 0$, $G(\omega) \to \frac{F_0}{k}$ | displacement proportional to force

$\uparrow$ constant input

4) practical resonance occurs ~~on~~ at $\omega = \omega_{max}$, where $G(\omega_{max})$ is a maximum

find max of G:

$$\frac{dG}{d\omega} = \frac{-F_0}{2(\dots)^{3/2}} \left[2(k-m\omega^2)(-2m\omega) + 2c^2\omega \right] = 0.$$

$\uparrow$ never 0

$$2m(k-m\omega^2) = c^2$$

$$\Rightarrow \omega_{max}^2 = \frac{k}{m} - \frac{c^2}{2m^2}.$$

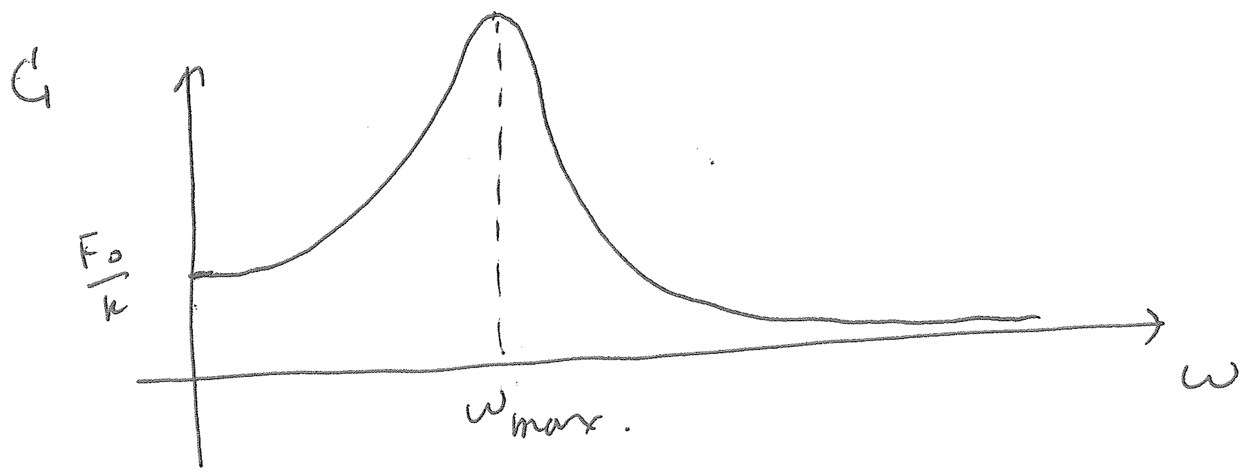
letting $\frac{k}{m} = \omega_0^2$ as before

$$\boxed{\omega_{max}^2 = \omega_0^2 - \frac{c^2}{2m^2}}$$

i) if c is small, $\omega_{max} \approx \omega_0$.

ii) if $\omega_0^2 - \frac{c^2}{2m^2} < 0$, ω_{max} does not exist, $\frac{dG}{d\omega} < 0$ for all ω.

(too much damping).



application to radio tuning.

simple radio uses RLC circuit as a band-pass filter (picks out a specific frequency out of a range of frequencies).

For RLC circuit (p. M5)

$$L I'' + r I' + \frac{1}{c} I = \frac{dV}{dt}$$

correct in notes

$m \to L$, $c \to r$, $k \to \frac{1}{c}$.

so $$\omega_{max}^2 = \frac{1}{LC} - \frac{r^2}{2L^2}.$$

now imagine we have many signals:

$$L I'' + r I' + \frac{1}{c} I = \sum_n A_n \cos(\omega_n t)$$

then
$$I = \cancel{I_h} + I_p$$

$$I_p = \sum_n \underbrace{G(\omega_n)}_{} \cos(\omega_n t - \alpha_n)$$

if the filter is "sharp"

can ~~pick out~~ isolate/amplify desired frequency ω_j

by "tuning" capacitance C such

that

$$\omega_{max} = \frac{1}{LC} - \frac{r^2}{2L^2} = \omega_j$$

so then RLC circuit filters out

all other frequencies.