Laplace Transforms (4.1)

- used to solve diff. eqn's by transforming the system to an algebraic system

  ODE → ~~algeb~~ algebraic

  PDE → ODE

- defn: the Laplace transform of $f(t)$ is defined as

$$\mathcal{L}(f(t)) = \underbrace{\int_0^\infty e^{-st} f(t)\,dt}_{\text{integrate out the } t} = F(s).$$

- convergence of integral may depend on s.

Ex (p. 267).

$$\mathcal{L}(1) = \int_0^\infty e^{-st}(1)\,dt = \frac{-1}{s} e^{-st}\Big|_0^\infty$$

$$= \frac{1}{s} \quad \text{as long as} \quad s > 0$$

if $s \leq 0$, then $e^{-st} \not\to 0$ as $t \to \infty$. so clearly integral converges only for $s > 0$

Ex (p. 267).

$$\mathcal{L}(e^{at}) = \int_0^\infty e^{-st} e^{at} \, dt$$

$$= \int_0^\infty e^{-(s-a)t} \, dt. = \frac{-1}{s-a} e^{-(s-a)t} \Big|_0^\infty$$

$$= \frac{-1}{s-a}(0-1) = \frac{1}{s-a} \quad ; \quad s > a.$$

Ex (p. 268)

$$\mathcal{L}(t^a) = \int_0^\infty e^{-st} t^a \, dt$$

require $a > -1$ for convergence near $t=0$. but a can be as large as we want since $e^{-st} t^a \to 0$ as $t \to \infty$ for any $\underline{\underline{a}}$ as long as $\underline{\underline{s > 0}}$

sub $u = st$, $du = s\,dt$.

then

$$\mathcal{L}(t^a) = \int_0^\infty e^{-u} \left(\frac{u}{s}\right)^a \frac{du}{s}$$

$$= \frac{1}{s^{1+a}} \int_0^\infty e^{-u} u^a \, du.$$

define $\Gamma(a+1) \equiv \int_0^\infty e^{-u} u^a \, du \quad a > -1.$

↑ Gamma function.

then

$$L(t^a) = \frac{\Gamma(a+1)}{s^{1+a}} \quad a > -1, \ s > 0$$

note: for $a > 0$, $\Gamma(a+1) = a\Gamma(a)$.

to show this, integrate by parts.

$$\Gamma(a+1) = \int_0^\infty e^{-u} u^a \, du = -e^{-u} u^a \Big|_0^\infty + \int_0^\infty e^{-u} a u^{a-1} du$$

$$= a \underbrace{\int_0^\infty e^{-u} u^{a-1} du}_{\text{immediate that } \Gamma(0) = \infty} = a\Gamma(a)$$

if a is an integer,

$$\Gamma(a+1) = a\Gamma(a) = a\Gamma(a-1+1)$$

$$= a(a-1)\Gamma(a-1)$$

$$= a(a-1)\Gamma(a-2+1)$$

$$= a(a-1)(a-2)\Gamma(a-2)$$

$$\vdots$$

$$= a(a-1)(a-2)\cdots(2)\underbrace{\Gamma(1)}_{\Gamma(1) = 1.}$$

$$= a!$$

$$\Rightarrow \Gamma(a+1) = a \quad \text{for } a \text{ integer} > 0.$$

$$\Rightarrow \mathcal{L}(t^n) = \frac{n!}{s^{1+n}}, \quad n \text{ integer} > 0.$$

$$s > 0.$$