Linearity of Laplace transforms

(4.1)

$$\mathcal{L}\left(af(t) + bg(t)\right) = a\mathcal{L}(f) + b\mathcal{L}(g).$$

proof:

$$\mathcal{L}\left(af(t)+bg(t)\right) = \int_0^\infty e^{-st}\left[af(t)+bg(t)\right]dt$$

$$= \int_0^\infty e^{-st} af(t)\,dt + \int_0^\infty b e^{-st} g(t)\,dt$$

$$= aF(s) + bG(s) \quad \blacksquare$$

Ex compute $\mathcal{L}(\cosh kt)$ $\quad k > 0$.

$$\cosh kt = \frac{e^{kt} + e^{-kt}}{2}.$$

$$\mathcal{L}(\cosh kt) = \frac{1}{2}\mathcal{L}(e^{kt}) + \frac{1}{2}\mathcal{L}(e^{-kt})$$

$\mathcal{L}(e^{at}) = \frac{1}{s-a}$

$$\frac{1}{2}\left[\frac{1}{s-k} + \frac{1}{s+k}\right]$$

$$= \frac{s}{s^2-k^2} \qquad s > k.$$

$$\int_0^\infty e^{-st} e^{kt}\,dt = \int_0^\infty e^{-(s-k)t}\,dt.$$

$$\mathcal{L}(\sinh kt) = \frac{k}{s^2-k^2}$$

Ex

$$\mathcal{L}(\cos kt) = \mathcal{L}\left(\frac{e^{ikt}+e^{-ikt}}{2}\right).$$

$$= \frac{1}{2}\mathcal{L}(e^{ikt}) + \frac{1}{2}\mathcal{L}(e^{-ikt})$$

$$= \frac{1}{2}\,\frac{1}{s-ik} + \frac{1}{2}\,\frac{1}{s+ik}$$

$\int_0^\infty e^{-st}\cos kt\,dt$

$$= \frac{s}{s^2+k^2} \qquad s > 0$$

$$\mathcal{L}(\sin kt) = \frac{k}{s^2+k^2}$$

EM

Ex

$$\mathcal{L}(e^{at}\cos kt) = \mathcal{L}\left(\frac{1}{2}\left(e^{(a+ik)t} + e^{(a-ik)t}\right)\right)$$

$$= \frac{1}{2}\left[\frac{1}{s-a-ik} + \frac{1}{s-a+ik}\right]$$

$$= \frac{1}{2}\left[\frac{(s-a)+ik + (s-a)-ik}{(s-a)^2+k^2}\right]$$

$$= \frac{s-a}{(s-a)^2+k^2} \qquad s>a$$

note: this is $\mathcal{L}(\cos kt)$ with $s-a$ subbed for s, because

$$\mathcal{L}(e^{at}\cos kt) = \int_0^\infty e^{-st} e^{at}\cos kt\,dt$$

$$= \int_0^\infty e^{-(s-a)t}\cos kt\,dt.$$

$$\mathcal{L}(e^{at}\sin kt) = \frac{k}{(s-a)^2+k^2}$$

Inverse Laplace transform

can calculate inverse explicitly
by computing some integral (online)
for now, use Laplace table

EX

$$\mathcal{L}^{-1}\left(\frac{1}{s^3}\right) = ?$$

we know $\boxed{\mathcal{L}(t^n) = \frac{n!}{s^{1+n}}}$

$$\mathcal{L}^{-1}\left(\frac{1}{s^3}\right) = \mathcal{L}^{-1}\left(\frac{1}{s^{1+2}}\right) = \frac{1}{2!}\boxed{\mathcal{L}^{-1}\left(\frac{2!}{s^{1+2}}\right)}$$

$$= \frac{1}{2!}t^2$$

Ex $\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) = ?$ e^{-2t}

$$\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) = \mathcal{L}^{-1}\left(\frac{1}{s-(-2)}\right) = e^{-2t}$$

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}$$

$$\Rightarrow e^{at} = \mathcal{L}^{-1}\left(\frac{1}{s-a}\right)$$

Ex $\mathcal{L}^{-1}\left(\frac{2}{s^2+9}\right)$; use

$$\mathcal{L}(\sin kt) = \frac{k}{s^2+k^2}$$

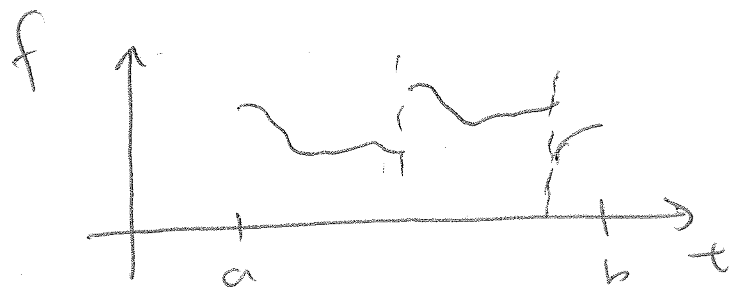
$$\mathcal{L}^{-1}\left(\frac{2}{s^2+9}\right) = \frac{1}{3/2}\mathcal{L}^{-1}\left(\frac{2 \cdot 3/2}{s^2+9}\right)$$

$$= \frac{2}{3}\mathcal{L}^{-1}\left(\frac{3}{s^2+9}\right) = \frac{2}{3}\sin 3t$$

Piecewise cont's functions

a piecewise continuous fcn $f(t)$ on $[a,b]$ is

1) continuous over a finite number of subintervals in $[a,b]$
2) has finite limit at t approaches endpoints of each subinterval.

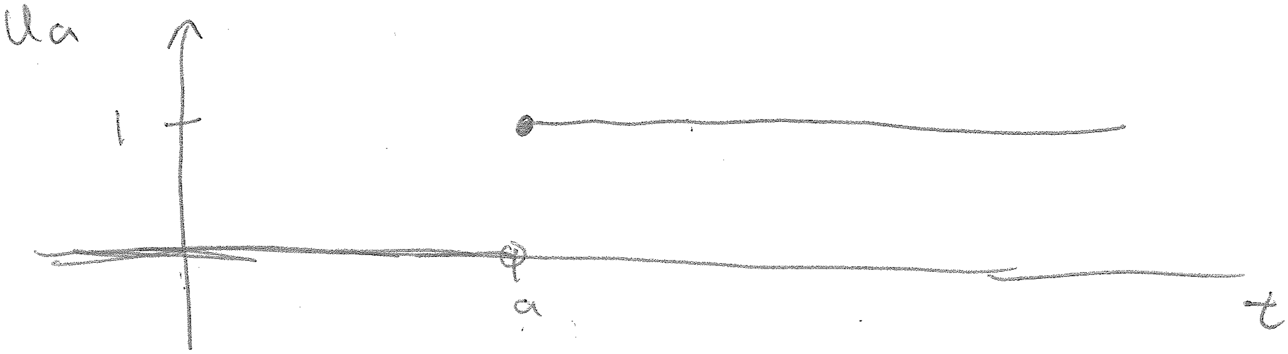


piecewise continuous.

$f(t) = \frac{1}{t}$ is not pwc on any interval containg $t=0$.

simplest pwc fcn is

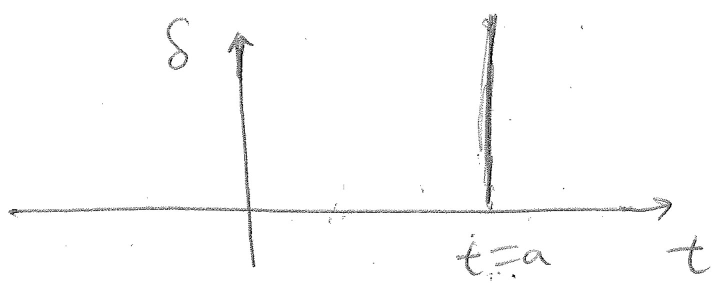
$$u_a(t) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$$



called Heaviside step function

integral of δ-function centered at

$t = a$



$$\int_{-\infty}^{t} \delta(t-a)\,dt = \begin{cases} 0 & t < a \\ 1 & t > a \end{cases}$$

$$\mathcal{L}(u_a(t)) = \int_0^\infty e^{-st} u_a(t)\,dt.$$

$$= \int_a^\infty e^{-st} \cdot 1 \, dt = \frac{-1}{s} e^{-st} \Big|_a^\infty$$

$$= 0 + \frac{1}{s} e^{-as} \qquad s > 0.$$

Existence

$\mathcal{L}(f(t))$ exists if $\lim_{b\to\infty} \int_0^b e^{-st} f(t)\,dt.$

conditions sufficient for convergence:

suppose that

1) $f(t)$ is pwc on $0 < t \le A$ for any $A > 0$.

2) $|f(t)| \le M e^{ct}$ $t \ge A$ for some c

then

$$\int_0^\infty e^{-st} f(t)\,dt = \overset{\text{finite}}{\boxed{\int_0^A e^{-st} f(t)\,dt}} + \int_A^\infty e^{-st} f(t)\,dt.$$

now

$$\int_A^\infty f(t)\, e^{-st}\,dt \le \int_A^\infty |f(t)|\, e^{-st}\,dt$$

$$\le \int_A^\infty M e^{ct} e^{-st}\,dt = M \int_A^\infty e^{-(s-c)t}\,dt$$

pick $s > c$

$\Rightarrow \int_0^\infty e^{-st} f(t)\,dt$ converges as long as $s > c$.

such functions $f(t)$ are said to be of <u>exponential order</u>.

(P15)

t^n $n > 0$, $\cos kt$, e^{kt}

all exp. order

t^t, e^{t^2} $\quad e^{t^2}/e^{ct} = e^{t^2 - ct}$

as $t \to \infty$, cannot pick c to make

$e^{t^2 - ct} \to 0$