uniquess of Laplace transforms

If $F(s) = G(s)$, then $f(t) = g(t)$.

Solutions of 2^{nd} order constant coeff IVP's (4.2)

here we solve

$$ay'' + by' + cy = f(t)$$

by applying $\mathcal{L}(\cdot)$ to both sides of the equation → solve algebraic equation for $\mathcal{L}(y(t))$, then invert for $y(t)$.

to do so, we need to compute

$$\mathcal{L}(y'') \text{ and } \mathcal{L}(y')$$

$$\mathcal{L}(y') = \int_0^\infty e^{-st} \frac{dy}{dt}\, dt \overset{\text{ibp.}}{=} e^{-st} y \Big|_0^\infty + s \int_0^\infty e^{-st} y \, dt.$$

$$= 0 - y(0) + s\mathcal{L}(y) \qquad \Big| \; Y(s) = \mathcal{L}(y)$$

$$\Rightarrow \mathcal{L}(y') = sY(s) - y(0)$$

$$\mathcal{L}(y'') = \int_0^\infty e^{-st} y''\,dt = e^{-st} y' \Big|_0^\infty + s \underbrace{\int_0^\infty e^{-st} y'\,dt}_{\mathcal{L}(y')}$$

$$= 0 - y'(0) + s\left[sY(s) - y(0)\right]$$

$$= s^2 Y(s) - s y(0) - y'(0).$$

require y'' piecewise cont's

y' is cont's

y is C^1 smooth.

if y is C^{n-1} smooth,

$$\mathcal{L}(y^{(n)}) = s^n Y(s) - s^{n-1} y(0) + \dots + s y^{(n-2)}(0) - y^{(n-1)}(0)$$

<u>Ex</u> Solve $y'' - y' - 2y = 0$, $y(0) = 1$, $y'(0) = 0$

~~cut~~ from C.E., $y = \frac{2}{3} e^{-t} + \frac{1}{3} e^{2t}$.

with Laplace transforms | $Y(s) = \mathcal{L}(y)$

$$\mathcal{L}(y'') - \mathcal{L}(y') - 2\mathcal{L}(y) = 0 \quad \text{(by linearity)}$$

$$s^2 Y(s) - s y(0) - \cancel{y'(0)} - (s Y(s) - y(0)) - 2 Y(s) = 0$$

$$(s^2 - s - 2) Y(s) - s + 1 = 0$$

$$Y(s) = \frac{s-1}{(s-2)(s+1)}$$

$\mathcal{L}^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$.

$$Y(s) = \frac{A}{s-2} + \frac{B}{s+1} \quad (\text{p.f.e})$$

$$A(s+1) + B(s-2) = s-1$$

$\underline{s=-1}$: $B(-3) = -2 \Rightarrow B = 2/3$

$\underline{s=2}$: $3A = 1 \Rightarrow A = 1/3$.

$$Y(s) = \frac{1/3}{s-2} + \frac{2/3}{s+1}$$

$$y(t) = \mathcal{L}^{-1}(Y(s)) = \frac{1}{3}\mathcal{L}^{-1}\left(\frac{1}{s-2}\right) + \frac{2}{3}\mathcal{L}^{-1}\left(\frac{1}{s+1}\right)$$

$$= \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}.$$

EY

$$y'' + y = \sin 2t \quad y(0) = 2, \; y'(0) = 1.$$

apply $\mathcal{L}(\cdot)$ to ODE:

$\mathcal{L}(\sin kt) = \dfrac{k}{s^2+k^2}$

$$\mathcal{L}(y'') + \mathcal{L}(y) = \mathcal{L}(\sin 2t)$$

$$s^2 Y(s) \boxed{- s\,y(0)} \boxed{- y'(0)} + Y(s) = \frac{2}{s^2+4}$$

$$(s^2+1)Y(s) - 2s - 1 = \frac{2}{s^2+4}$$

$$(s^2+1)Y(s) = 1 + 2s + \frac{2}{s^2+4}$$

$$= \frac{2s^3 + s^2 + 8s + 6}{s^2+4}.$$

$$Y(s) = \frac{2s^3 + s^2 + 8s + 6}{(s^2+1)(s^2+4)}$$

PFE: $$Y(s) = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

so

$$2s^3 + s^2 + 8s + 6 = (As+B)(s^2+4) + (Cs+D)(s^2+1)$$

set $s=i$: $-2i - 1 + 8i + 6 = (Ai+B)(-1+4)$

$$\boxed{6}\,i + 5 = \boxed{3A}\,i + 3B$$

imag : $6 = 3A \Rightarrow A = 2.$

real : $5 = 3B \Rightarrow B = 5/3.$

set $s=2i$

$$-\cancel{16i} - 4 + \cancel{16i} + 6 = (2iC + D)(-3)$$

$$2 = -6iC - 3D$$

$$\Rightarrow C = 0, \quad D = -2/3.$$

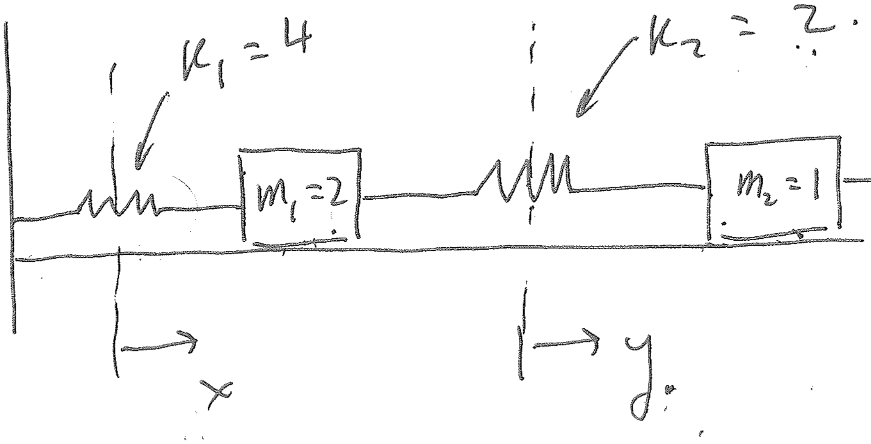
$$Y(s) = \frac{2s + 5/3}{s^2+1} - \frac{2}{3}\,\frac{1}{s^2+4}$$

$$\mathcal{L}(\cos kt) = \frac{s}{s^2+k^2}, \quad \mathcal{L}(\sin kt) = \frac{k}{s^2+k^2}.$$

$$Y(s) = 2\boxed{\frac{s}{s^2+1}} + \frac{5}{3}\frac{1}{s^2+1} - \frac{1}{3}\frac{2}{s^2+4}$$

$$= 2\cos t + \frac{5}{3}\sin t - \frac{1}{3}\sin 2t.$$

<u>EX</u> from book (4.2) - Laplace transform for a system of equations



$f(t) = 40\sin 3t.$

$x(0) = x'(0) = y(0) = y'(0) = 0$

$$2x'' = -4x + 2(y-x)$$

$$= -6x + 2y.$$

$$1y'' = -2(y-x) + 40\sin 3t.$$

apply $\mathcal{L}(\cdot)$ to both equations.

$$2s^2 X(s) = \cancel{0} - 6X + 2Y$$

$$s^2 Y(s) = 2X - 2Y + 40 \cdot \frac{3}{s^2+9}$$

algebraic equation for X and Y (linear)

$$X(s) = \frac{120}{(s^2+1)(s^2+4)(s^2+9)}$$

$$Y(s) = \frac{120(s^2+3)}{(s^2+1)(s^2+4)(s^2+9)}$$

now invert $x(t) = \mathcal{L}^{-1}(X)$, $y(t) = \mathcal{L}^{-1}(Y)$.

$$X(s) = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4} + \frac{Es+F}{s^2+9}$$

so

$$(As+B)(s^2+4)(s^2+9) + (Cs+D)(s^2+1)(s^2+9) + (Es+F)(s^2+1)(s^2+4) = 120$$

set $s=i$

$$(Ai+B)(3)(8) = 120 \Rightarrow A=0$$
$$B=5$$

set $s=2i$

$$(2iC+D)(-3)(5) = 120 \Rightarrow C=0$$
$$D=-8.$$

set $s=3i$

$$(3iE+F)(-8)(-5) = 120$$
$$\Rightarrow E=0$$
$$F=3.$$

$$X(s) = \frac{5}{s^2+1} - \frac{8}{s^2+4} + \frac{3}{s^2+9}.$$

$$x(t) = 5\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) - 4\mathcal{L}^{-1}\left(\frac{2}{s^2+4}\right) + \mathcal{L}^{-1}\left(\frac{3}{s^2+9}\right)$$

$$x(t) = \underbrace{5\sin t - 4\sin 2t}_{\text{natural freq.}} + \sin 3t.$$

forcing frequency.

$$Y(s) = \frac{10}{s^2+1} + \frac{8}{s^2+4} - \frac{18}{s^2+9}.$$

$$y(t) = 10\sin t + 4\sin 2t - 6\sin 3t.$$

can solve also by writing as a fourth order ODE

(system of 2 second ODE's $\rightarrow$ 1 4th order eqn).

$$2x'' = -6x + 2y \Rightarrow y = x'' + 3x.$$

$$y'' = x^{(iv)} + 3x''$$

then from

$$y'' = 2x - 2y + 40\sin 3t.$$

$$x^{(iv)} + 3x'' - 2x + 2(x'' + 3x)$$

$$= 40\sin 3t.$$

$$x^{(iv)} + 5x'' + 4x = 40 \sin 3t.$$

apply $\mathcal{L}(\cdot)$ ... etc.

In general, for second order constant coeff ODE

$$mx'' + cx' + kx = f(t), \quad x(0) = \alpha, \quad x'(0) = \beta$$

apply $\mathcal{L}(\cdot)$

$$m\left[s^2 X - s x(0) - x'(0)\right] + c\left[sX(s) - x(0)\right] + kX(s) = F(s).$$

$$(ms^2 + cs + k)X(s) - \left[(ms+c)x(0) + m x'(0)\right] = F(s).$$

define

$$Z(s) = ms^2 + cs + k$$

$$I(s) = ms\,x(0) + m x'(0) + c\,x(0)$$

then

$$X(s) = \frac{F(s)}{Z(s)} + \frac{I(s)}{Z(s)}$$

transient term contains info about IC's.

dominates in large time if $c > 0$

$\frac{1}{Z(s)}$ is called the transfer function.

so the method of Laplace transforms comes down to inverting

$$\frac{F(s) + I(s)}{Z(s)}.$$

Laplace transforms can also be used to solve integral equation (equations involving integrals)

transform of integral

Let $f(t)$ be piecewise cont's, and exp. order, then

$$\mathcal{L}\left\{ \int_0^t f(\tau)\,d\tau \right\} = \frac{1}{s} F(s)$$

proof:

let $g(t) = \int_0^t f(t)\,dt.$

want to show

$$\mathcal{L}(g(t)) = G(s) = \frac{F(s)}{s.}$$

$$\mathcal{L}(g') = s\,G(s) - \cancel{g(0)} \to 0 \text{ since } \int_0^0$$

$$\Rightarrow \mathcal{L}(g') = s\,G(s).$$

$$\mathcal{L}(f) = s\,G(s) \Rightarrow G(s) = \frac{F(s)}{s}$$

□

Ex (p. 285).

compute $\mathcal{L}^{-1}(G(s))$ where

$$G(s) = \frac{1}{s^2(s-a)}$$

let $F(s) = \dfrac{1}{s-a} \Rightarrow \mathcal{L}^{-1}(F(s)) = e^{at}$

then $G(s) = \dfrac{F(s)}{s^2} = \dfrac{1}{s}\,\dfrac{F(s)}{s}$

$$\mathcal{L}^{-1}\left(\frac{F(s)}{s}\right) = \int_0^t e^{at}\,dt. = \frac{1}{a}(e^{at}-1)$$

$$\mathcal{L}^{-1}\left(\frac{F(s)}{s^2}\right) = \mathcal{L}^{-1}\left(\frac{1}{s}\,\frac{F(s)}{s}\right)$$

$$= \int_0^t \frac{1}{a}(e^{at}-1)\,dt = \frac{e^{at}}{a^2} - \frac{t}{a} - \frac{1}{a^2}.$$

Translation rule and partial fractions (4.3)

translation in the s-variable

let $\mathcal{L}(f(t)) = F(s) = \int_0^\infty e^{-st} f(t)\,dt.$

then

$$\mathcal{L}(e^{at} f(t)) = \int_0^\infty e^{-st} e^{at} f(t)\,dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\,dt.$$

$$= F(s-a)$$

$$\Rightarrow \mathcal{L}(e^{at} f(t)) = F(s-a)$$

or

$$\mathcal{L}^{-1}(F(s-a)) = e^{at} f(t)$$

Ex

$\mathcal{L}(te^{at}) = ?$

$\mathcal{L}(t) = \frac{1}{s^2} \Rightarrow \mathcal{L}(te^{at}) = \frac{1}{(s-a)^2}$

EX compute $\mathcal{L}(t \sin kt)$

use $\mathcal{L}(\sin kt) = \dfrac{k}{s^2+k^2}$.

$$\mathcal{L}(t \sin kt) = \mathcal{L}\left\{ t \left(\frac{e^{ikt} - e^{-ikt}}{2i} \right) \right\}$$

$$= \frac{1}{2i} \mathcal{L}(t e^{ikt}) - \frac{1}{2i} \mathcal{L}(t e^{-ikt})$$

$$= \frac{1}{2i} \frac{1}{(s-ik)^2} - \frac{1}{2i} \frac{1}{(s+ik)^2}$$

$\vdots$

$$= \frac{2sk}{(s^2+k^2)^2}$$

partial fractions

let $R(s) = \dfrac{P(s)}{Q(s)}$, where

degree (P) < degree (Q).

rule 1

if $Q(s)$ has a zero at $s=a$ of multiplicity n (i.e., $Q(s)=(s-a)^n q(s)$ where $q(a) \neq 0$)

then the portion of the p.f.e. corresponding to $s-a$ is

$$\frac{A_1}{s-a} + \frac{A_2}{(s-a)^2} + \cdots + \frac{A_n}{(s-a)^n}.$$

rule 2

if $Q(s) = \left((s-a)^2+b^2\right)^n q(s)$, where $q(a \pm ib) \neq 0$, then ~~corre~~ the portion of the p.f.e corresponding to $(s-a)^2+b^2$ is

(P33)

$$\frac{A_1 s + B_1}{(s-a)^2+b^2} + \frac{A_2 s + B_2}{((s-a)^2+b^2)^2} + \cdots$$

$$+ \frac{A_n s + B}{((s-a)^2+b^2)^n}$$

$$\frac{A_1 s + B_1}{(s-a)^2+b^2} = \frac{A_1 (s-a)}{(s-a)^2+b^2} + \frac{B + A_1 a}{(s-a)^2+b^2}$$

$(\ )e^{-at} \cos bt$ $\qquad$ $(\ )e^{-at} \sin bt.$

rule