last time

$$y'' + 9y = 0 \qquad \text{C.E.} \quad r^2 + 9 = 0$$

$r = \pm 3i$

$y_1 = e^{i3x}$, $y_2 = e^{-i3x}$

notice : $y_2 = \overline{y_1}$ $(i \to -i)$

so $y = c_1 e^{i3x} + c_2 e^{-i3x}$

can also write

$$y = c_1 \,\mathrm{Re}(y_1) + c_2 \,\mathrm{Im}(y_1)$$

the reason is $\mathrm{Re}(y_1) = \dfrac{y_1 + y_2}{2}$

$\mathrm{Im}(y_1) = \dfrac{y_1 - y_2}{2i}$

<u>Ex</u> $y'' + y' + y = 0$

C.E. $r^2 + r + 1 = 0$ (from guess $y = e^{rx}$)

so $r_{1,2} = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

so $y = c_1 e^{\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)x} + c_2 e^{\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)x}$

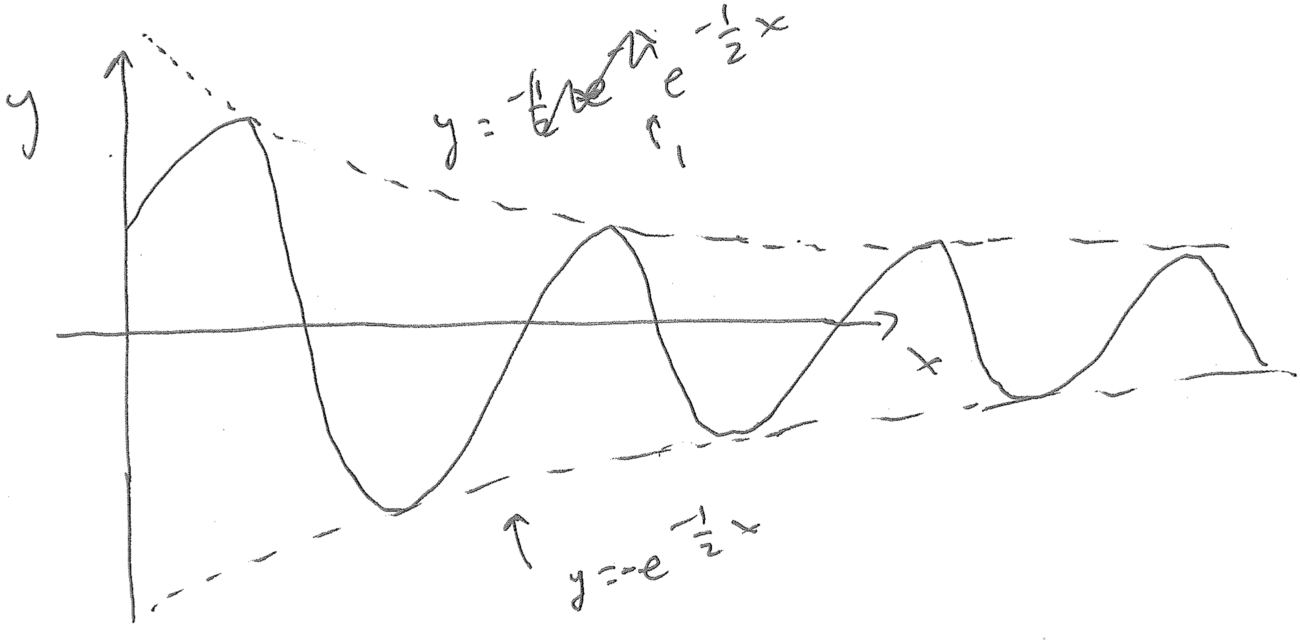
notice: $y_1 = e^{\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)x}$

$= e^{-\frac{1}{2}x} e^{i\frac{\sqrt{3}}{2}x}$

$= e^{-\frac{1}{2}x}\left(\underbrace{\cos\frac{\sqrt{3}}{2}x}_{\text{real part}} + i\underbrace{\sin\frac{\sqrt{3}}{2}x}_{\text{imag. part}}\right)$

therefore can also write general solution as

$$y = \underbrace{e^{-\frac{1}{2}x}}_{\substack{\text{amplitude} \\ \text{decay (envelope)}}} \underbrace{\left[c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right]}_{\text{oscillations}}$$



so for <u>case 2</u>, $b^2 - 4ac < 0$,

define $\gamma = \frac{-b}{2a}$, $\omega = \frac{1}{2a}\sqrt{4ac - b^2}$

↑ real

a general solution can be written as

$$y = e^{\gamma x}\left[c_1 \cos\omega x + c_2 \sin\omega x\right]$$

=

$$ar^2 + br + c = 0$$

$$r = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \quad ; \quad b^2 - 4ac < 0$$

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$$r = \frac{-b}{2a} \pm \frac{1}{2a}\sqrt{(-1)(4ac-b^2)}$$

$$= \frac{-b}{2a} \pm \frac{1}{2a}\sqrt{-1}\sqrt{4ac-b^2}$$

$$= \frac{-b}{2a} \pm \frac{i}{2a}\sqrt{4ac-b^2}$$

$$= \gamma \pm i\omega$$

then the two solutions would be

$$y_1 = e^{(\gamma+i\omega)x} \qquad y_2 = e^{(\gamma-i\omega)x}$$

$$y_1 = e^{\gamma x} e^{i\omega x} = e^{\gamma x}[\underbrace{\cos \omega x}_{\text{real}} + i \underbrace{\sin \omega x}_{\text{imag.}}]$$

$$\Rightarrow y = e^{\gamma x}[c_1 \cos \omega x + c_2 \sin \omega x].$$

the last case is $\sqrt{b^2-4ac} = 0$

(Case 3)

$$r_{1,2} = \frac{-b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a} = \frac{-b}{2a}.$$

this yields only one soln $y_1 = e^{\frac{-b}{2a}x}$

need to find another l.i. soln

↳ use reduction of order.

<u>Ex</u> solve

$$y'' + 4y' + 4y = 0 \qquad (6)$$

C.E. $r^2 + 4r + 4 = 0$

$(r+2)^2 = 0 \Rightarrow r = -2, -2$

so we have

$$y_1 = e^{-2x}.$$

look for $y_2(x)$ in the form of

$$y_2 = \underbrace{v(x)}_{\text{"guess" for reduction of order}} y_1 = v(x)e^{-2x} \qquad (7)$$

and derive ODE for $v(x)$ (will see it is much simpler than original ODE). substitute (7) into (6).

so we compute:

$$y_2' = v' e^{-2x} - 2ve^{-2x}$$

$$= e^{-2x}(v' - 2v)$$

$$y_2'' = v'' e^{-2x} - 2v' e^{-2x}$$
$$- 2\left[v' e^{-2x} - 2ve^{-2x}\right].$$

$$= e^{-2x}\left[v'' - 4v' + 4v\right].$$

put into (6) and cancel e^{-2x} (never 0):

$$v'' \cancel{-4v'} + \cancel{4v} + 4\left[\cancel{v'} - \cancel{2v}\right] + \cancel{4v} = 0$$

$$\Rightarrow \quad v'' = 0$$ (much simpler than original)

$\Rightarrow \quad v' = a_1$

$v = a_1 x + a_2$

do we need this?

can take $a_2 = 0$ w log.

$a_1 = 1$ wlog

$y_1 = e^{-2x}, \quad y_2 = (a_1 x + a_2) e^{-2x}.$

so general soln is

$$y = c_1 e^{-2x} + c_2 (a_1 x + a_2) e^{-2x}$$

$$= (c_1 + a_2 c_2) e^{-2x} + a_1 c_2 x e^{-2x}$$

$$= A e^{-2x} + \cancel{B e^{-2x}} \; B x e^{-2x}$$

summarize results:

for 2^{nd} order ODE with constant coeff's (homogeneous)

$$a y'' + b y' + c y = 0$$

let r_1, r_2 be roots of C.E.

$$ar^2+br+c=0.$$

<u>case 1</u> r_1, r_2 real, disetinct $(b^2-4ac>0)$

$$y_1 = e^{r_1 x}, \quad y_2 = e^{r_2 x}$$

so $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$ is general soln.

<u>case 2</u> : $r_1 = \gamma + i\omega$, $r_2 = \gamma - i\omega$

$(b^2-4ac<0)$

always be complex conjugates if a, b, c are real

then $y_1 = e^{\gamma x}\cos\omega x$, $y_2 = e^{\gamma x}\sin\omega x$.

$\Rightarrow y = e^{\gamma x}[c_1\cos\omega x + c_2\sin\omega x]$

is general soln

case 3

r_1, r_2 are equal $(b^2 - 4ac = 0)$.

$$y_1 = e^{r_1 x}, \quad y_2 = x e^{r_1 x}$$

from reduction of order.

$$y = c_1 e^{r_1 x} + c_2 x e^{r_1 x}$$

EX 3rd order, distinct roots

find ~~three di~~ three l.i. solns of

one derivative

$$y''' - 3y'' - 4y' + 12y = 0$$

hint $y_1 = e^{2x}$ is a solution

could use reduction of order to reduce to 2nd order ODE, or use synthetic or polynomial division on C.E.

procedure is the same: guess $y = e^{rx}$

So C.E is

$$r^3 - 3r^2 - 4r + 12 = 0$$

from the fact that $y = e^{2x}$ is a solution of ODE, $r-2$ must be a factor of LHS :

$$\begin{array}{r|rrrr} & 1 & -3 & -4 & 12 \\ -2 & & -2 & 2 & 12 \\ \hline & 1 & -1 & -6 & 0 \end{array}$$

$\Rightarrow \quad r^3 - 3r^2 - 4r + 12 = (r-2)(r^2 - r - 6)$

$\Rightarrow (r-2)(r-3)(r+2) = 0$

the roots $r_1 = 2$, $r_2 = 3$, $r_3 = -2$.

then the 3 l.i. solutions are :

$y_1 = e^{2x}$, $y_2 = e^{3x}$, $y_3 = e^{-2x}$

and

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wronskian:

$$W = \begin{array}{c} \\ y_1' \\ y_1'' \end{array} \begin{vmatrix} \overset{y_1}{e^{2x}} & e^{3x} & e^{-2x} \\ 2e^{2x} & 3e^{3x} & -2e^{-2x} \\ 4e^{2x} & 9e^{3x} & 4e^{-2x} \end{vmatrix} = -20e^{3x} \neq 0$$

will never be asked of you by me.

Ex (complex roots)

$y''' - y = 0 \Rightarrow$ guess $y = e^{rx}$

$r^3 - 1 = 0$

$r_1 = 1$

$r_1^3 = 1 = e^{i \cdot 0}$

$r_1 = e^{i0/3} = 1$

$r_2^3 = 1 = e^{i2\pi}$

$r_2 = e^{i2\pi/3}$

aside:

$y'' - y = 0$

$r^2 - 1 = 0$

$r = \pm 1$

$r^2 = 1 = e^{i \cdot 0}$

$r = e^{i0/2} = 1$

$r^2 = 1 = e^{i2\pi}$

$r = e^{i\pi} = -1$

(D39)

$r_3^3 = 1 = e^{i4\pi}$

$r_3 = e^{i4\pi/3}$

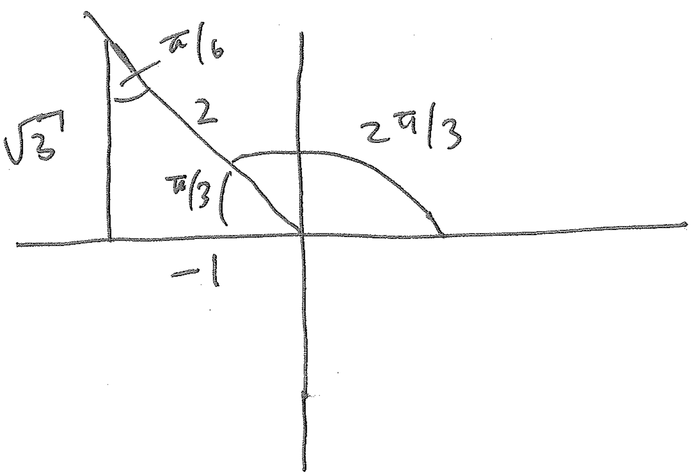
$r_4^3 = 1 = e^{i6\pi}$

$r_4 = e^{i2\pi} = 1 = r_1$

$r^2 = 1 = e^{i4\pi}$

$r = e^{i2\pi} = 1$

$r_2 = e^{i2\pi/3} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$

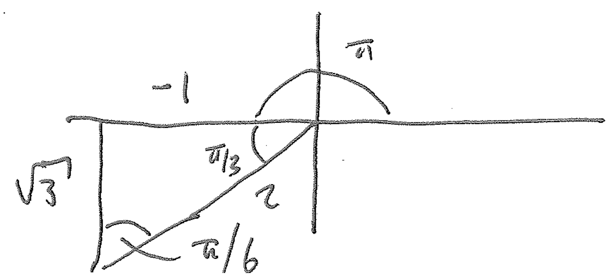


$\Rightarrow \cos\frac{2\pi}{3} = -\frac{1}{2}$

$\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$

$\Rightarrow r_2 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$

$r_3 = e^{i4\pi/3} = \cos 4\frac{\pi}{3} + i\sin 4\frac{\pi}{3}$



$$\cos\frac{4\pi}{3} = -\frac{1}{2}, \quad \sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2}.$$

$$r_3 = -\frac{1}{2} - \frac{i\sqrt{3}}{2} \quad (\text{complex conj. of } r_2)$$

so $y_1 = e^x$, $y_2 = e^{\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)x}$

$$y_3 = e^{\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)x}.$$

general solution

$$y_0 = c_1 e^x + e^{-\frac{1}{2}x}\left[c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right].$$

IC's: $y(t_0) = p_0$, $y'(t_0) = v_0$, $y''(t_0) = a_0$

$\hookrightarrow$ solve 3×3 system for c_1, c_2, c_3

<u>Ex</u> (repeated roots).

$$y''' + 3y'' + 3y' - y = 0$$

guess $y = e^{rx}$

$$r^3 - 3r^2 + 3r - 1 = 0$$

$$(r-1)^3 = 0 \Rightarrow r = 1, 1, 1$$

$$y_1 = e^x, \quad y_2 = xe^x, \quad y_3 = x^2 e^x$$

then $y = c_1 y_1 + c_2 y_2 + c_3 y_3$ is

general soln.

same for 4th order etc...

<u>Back to reduction of order</u>.

recall example: $y'' + 4y' + 4y = 0$

found one soln $y_1 = e^{-2x}$

<u>guessed</u> at form $y_2 = v(x) y_1 = v(x) e^{-2x}$

derived ODE for $v(x)$ that was simpler

than original ODE.

this idea of obtaining second solution

from the first also work for non-constant coeff's. ie.

$$y'' + p(x)\,y' + q(x)\,y = 0$$

Ex

$$y'' - \frac{x}{x-1}\,y' + \frac{1}{x-1}\,y = 0$$

find y_2, given $y_1 = e^x$

guess $y_2 = v(x)\,e^x$,

$$y_2' = v'e^x + ve^x$$

$$y_2'' = v''e^x + 2v'e^x + ve^x$$

sub into ODE:

$$\cancel{e^x}\left[v'' + 2v' + \underset{\smile}{v} - \frac{x}{x-1}(v' + v) + \boxed{\frac{1}{x-1}\,v}\right] = 0$$

always always: the zeroth derivative term will cancel

indeed: $\underset{\smile}{1} - \frac{x}{x-1}\ \boxed{+\frac{1}{x-1}} = \frac{x-1-x+1}{x-1} = 0$

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$$\Rightarrow \quad v'' + 2v' - \frac{x}{x-1} v = 0$$

$$v'' + \frac{x-2}{x-1} v' = 0$$

look like second order, but effectively it is a first order equation (difference between highest order deriv and lowest is $\underline{\underline{1}}$)

let $u = v'$, then

$$u' + \frac{x-2}{x-1} u = 0 \qquad \text{(sepeable, linear ...)}$$

first order equation for u

$$\frac{du}{dx} = -\left(\frac{x-2}{x-1}\right) u.$$

$$\int \frac{du}{u} = -\int \frac{x-2}{x-1} dx$$

$$\int \frac{x-2}{x-1} dx = \int \frac{x-1-1}{x-1} dx = \int 1 - \frac{1}{x-1} dx$$

$$= x - \log(x-1).$$

$$\Rightarrow \log u = -x + \log(x-1) + c.$$

$$u = c\left[e^{-x+\log(x-1)}\right].$$

$$= c\, e^{-x}(x-1)$$

require $v = \int u\, dx$

$$v = -x e^{-x} + c \qquad \text{(integration by parts)}$$

then

$$y_2 = v e^x = (-x e^{-x} + c)\, e^x$$

$c = 0$ wlog.

$$= -x$$

$$\Rightarrow \boxed{y_2 = x}$$