EX $\quad y'' - \frac{x}{x-1} y' + \frac{1}{x-1} y = 0$

$y_1 = e^x$

use reduction of order to find another l.i. solution

$y_2(x) = v(x)\, e^x$

$y_2' = v' e^x + v e^x$

$y_2'' = v'' e^x + v' e^x + v' e^x + v e^x$

$= v'' e^x + 2v' e^x + v e^x$

$$e^x \left[v'' + 2v' \cancel{+ v} - \frac{x}{x-1} \left(v' \cancel{+ v} \right) \cancel{+ \frac{1}{x-1} v} \right]$$

will always cancel

$$v'' + 2v' - \frac{x}{x-1} v' = 0$$

$$v'' + \frac{x-2}{x-1} v' = 0$$

looks like second-order, but effectively it is first order because the difference between the highest and lowest order derivatives is <u>one</u>

let $u = v'$

$$u' + \frac{x-2}{x-1} u = 0 \qquad \text{(separable, linear)}$$

$$\frac{du}{dx} = -\left(\frac{x-2}{x-1}\right) u$$

$$\int \frac{du}{u} = -\int \frac{x-2}{x-1}\, dx$$

$$\int \frac{x-2}{x-1}\,dx = \int \frac{x-1-1}{x-1}\,dx$$

$$= \int 1 - \frac{1}{x-1}\,dx$$

$$= x - \log(x-1)$$

$$\Rightarrow \log u = -x + \log(x-1) + c$$

$$u = C e^{-x+\log(x-1)}$$

$$= C e^{-x} e^{\log(x-1)}$$

$$= C e^{-x}(x-1) \qquad C = 1 \text{ wlog.}$$

now

$$v = \int u\,dx$$

integration by parts

$$= \int (x-1)e^{-x}\,dx = -(x-1)e^{-x} - \int (x-1)e^{-x}\,dx.$$

$$= -(x-1)e^{-x} + e^{-x} + c = -xe^{-x} \underset{\substack{\uparrow \\ c=0 \\ \text{wlog.}}}{+c}$$

finally,

$$y_2 = v(x) \underbrace{e^x}_{y_1}$$

$$= -xe^{-x}e^x = -x.$$

$$\Rightarrow \boxed{y_2 = x} \qquad y_2 = -x \qquad y_2 = 10x$$

$$y = \underbrace{a}e^x + \underbrace{b}e^{-x}$$

in general, for

$$y'' + p(x)y' + q(x)y = 0$$

given a solution $y_1(x)$, find $y_2(x)$ in the form

$$y_2(x) = v(x)\,y_1(x)$$

for $v(x)$ to be found.

$$y_2' = v' y_1 + v y_1'$$

$$y_2'' = v'' y_1 + 2v' y_1' + v y_1''$$

put into ODE.

$$v'' y_1 + 2v' y_1' + v y_1'' + p(x) [v' y_1 + v y_1']$$

$$+ q(x) v y_1 \quad = 0$$

$$v'' y_1 + 2v' y_1' + p(x) y_1 v' + v [y_1'' + p(x) y_1'$$
$$+ q(x) y_1] = 0$$

$y_1(x)$ is a solution of the ODE

$\Rightarrow$ term multiply v is 0

$$v'' y_1 + (2y_1' + p(x) y_1) v' = 0$$

$u = v'$

$$u' y_1 + (2y_1' + p(x) y_1) u = 0.$$

$$\boxed{u' = -\frac{(2y_1' + p(x) y_1)}{y_1} u}$$

separable equation, also linear.

then $v = \int u \, dx$

<u>an example with non-constant coeff's - Equidimensional equation.</u>

second order version:

$$x^2 y'' + a x y' + b y = 0.$$

also called Cauchy - Euler equation

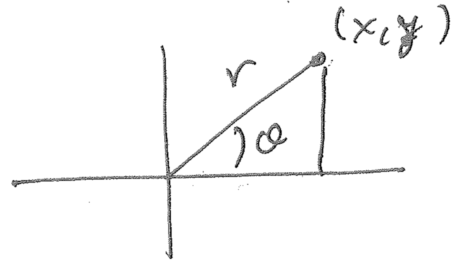
[comes in Laplace's equation in polar coordinates:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

D51

$$(x, y) \to (r, \theta)$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$



$$\hookrightarrow \quad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

$$u = R(r)\, e^{im\theta} \qquad \text{(separation of variables).}$$

then we have

$$R'' + \frac{1}{r} R' - \frac{m^2}{r^2} R = 0.$$

$$r^2 R'' + r R' - m^2 R = 0.$$

to solve, guess

$$y = x^r, \qquad (r \text{ is constant}).$$

$$y' = r x^{r-1}$$

$$y'' = r(r-1)\, x^{r-2}$$

sub into ODE and factor out x^r:

$$x^r \left[r(r-1) + ar + b \right] = 0$$

since $x \neq 0$,

$$r(r-1) + ar + b = 0.$$

$$r^2 + (a-1)r + b = 0$$

3 cases

1) $r = r_1, r_2$ with r_1, r_2 distinct and real

$$y = c_1 x^{r_1} + c_2 x^{r_2}$$

2) r_1, r_2 complex conjugates.

$$r_1 = \alpha + i\beta, \quad r_2 = \alpha - i\beta$$

$$x^{\alpha + i\beta} = x^{\alpha} x^{i\beta} = x^{\alpha} \left[e^{\log x} \right]^{i\beta}$$

$$= x^{\alpha} \{ e^{i\beta \log x}$$

$$= x^{\alpha} \left[\underbrace{\cos \beta \log x}_{\text{real part}} + i \underbrace{\sin \beta \log x}_{\text{imag.}} \right]$$

$$y = ~~\text{[scribble]}~~ x^{\alpha}\left[c_1 \cos \beta \log x + c_2 \sin \beta \log x\right]$$

3) $r_1 = r_2 = r$ (repeated roots)

$$y = c_1 x^r + \underbrace{c_2 x^r \log x}_{\text{from reduction of order}}$$

side note: can also make transformation

$$x = e^t \rightarrow y(x) = \phi(\log x) = \phi(t).$$

then obtain a constant coefficient equation for $\phi(t)$