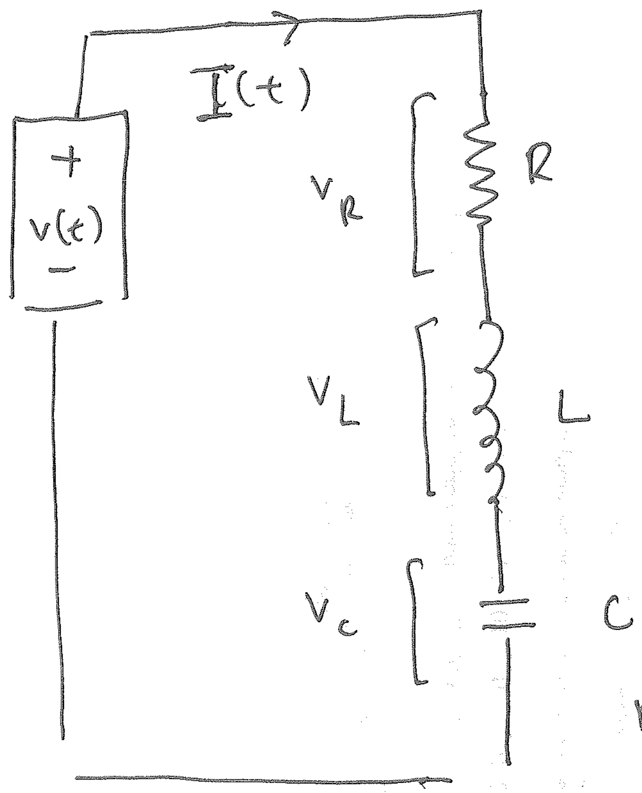
3rd example : RLC circuit

R ← resistor, L ← inductor, C ← capacitor



Kirchoff's Law:

$$V_R + V_L + V_C = V(t)$$

$$V_R \to \cancel{RI(t)}\ RI(t), \qquad V_L \to L\frac{dI}{dt}, \qquad V_C \to c\int_{-\infty}^{t} I(t)\,dt$$

$$rI(t) + L\frac{dI}{dt} + c\int_{-\infty}^{t} I(t)\,dt = V(t)$$

this is efficctively a second order ODE because $\int$ is a -1^{th} order derivative. Differentiate once to obtain

$$r\frac{dI}{dt} + L\frac{d^2 I}{dt^2} + cI(t) = \frac{dV}{dt}$$

if constant voltage, then $\frac{dV}{dt} = 0$

all have the form

$$ay'' + by' + cy = 0 \quad \text{(no external forcing)}$$

$b > 0$ is the physically relevant case or else $y \to \infty$ as $t \to \infty$.

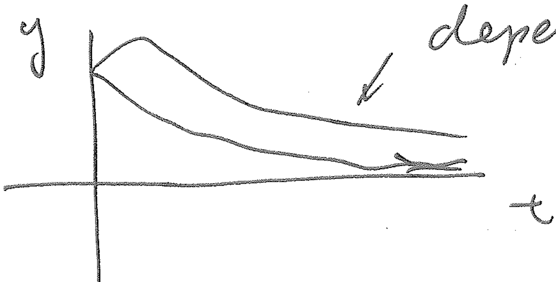
<u>3 cases as before</u>

① $b^2 - 4ac > 0$ (overdamped).

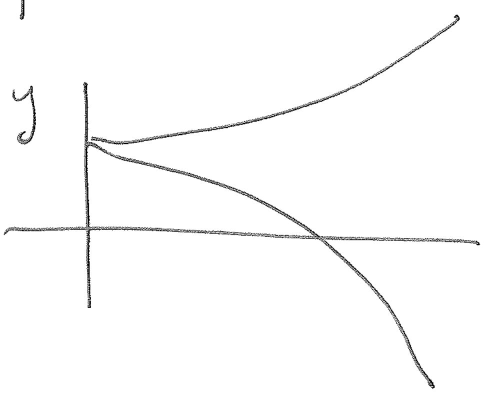
notice: $-b \pm \sqrt{b^2 - 4ac} < 0$ if $b > 0$

so $y \to 0$ as $t \to \infty$.

depending on IC's.

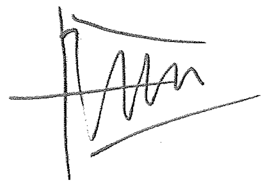


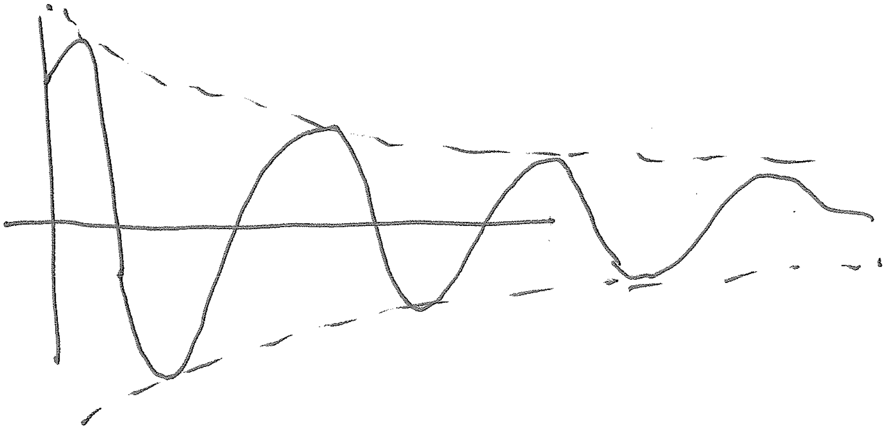
[if $b < 0$



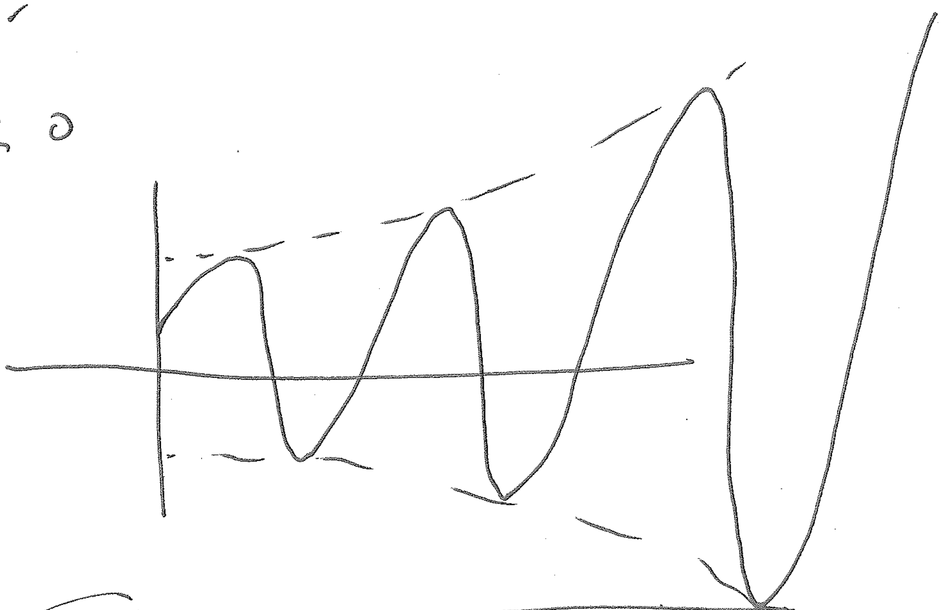
$|y| \to \infty$ as $t \to \infty$.
(not physical)

(2) $b^2 - 4ac < 0$ underdamped





if $b < 0$

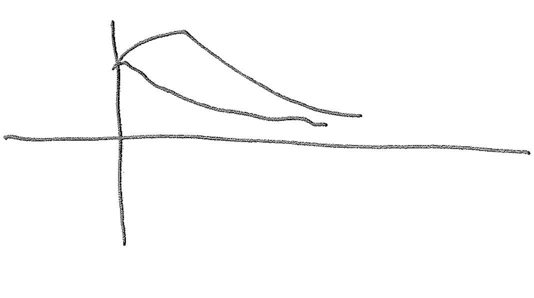


not physical

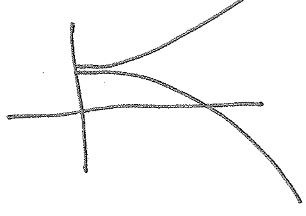
$$y = e^{-\gamma t}\left\{\boxed{c_1 \cos\omega t + c_2 \sin\omega t}\right\}$$

↑ envelope..

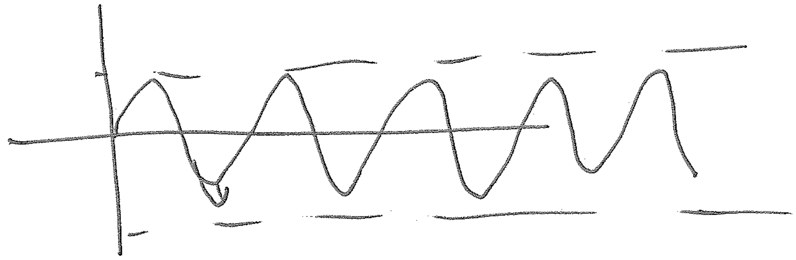
(3) $b^2 - 4ac = 0$ (critical damping)



if $b < 0$



$b = 0$ corresponds to undamped oscillations



constant amplitude

~~ph~~ Phase Amplitude representation

in the cases where the solution is oscillatory, information on the properties of the solution is more easily obtained from a phase-amplitude representation. That is we write

$$y = A\cos\omega t + B\sin\omega t$$

(A, B are from IC's)

in the form

$$y = C\cos(\omega t - \alpha)$$

$\uparrow C > 0$ $\qquad \uparrow 0 < \alpha < 2\pi.$

to find C and α ← phase angle.
↑ amplitude

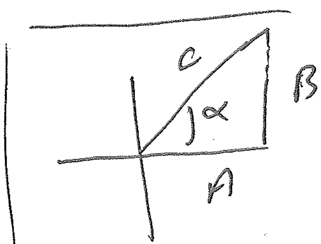
we write

$$C\cos(\omega t-\alpha) = C\left[\cos\omega t\cos\alpha + \underbrace{\sin\omega t\sin\alpha}\right].$$

like homework problem

the we equate to

$$y = A\cos\omega t + B\sin\omega t.$$



to get (matching coeff's)

$$\begin{cases} C\cos\alpha = A & \Rightarrow \quad \cos\alpha = A/C \\ C\sin\alpha = B & \Rightarrow \quad \sin\alpha = B/C \end{cases}$$

A, B known, C, α unknown

$$C^2\cos^2\alpha + C^2\sin^2\alpha = A^2+B^2$$

$$C^2\underbrace{\left[\cos^2\alpha + \sin^2\alpha\right]}_{1} = A^2+B^2$$

$$\boxed{C = \sqrt{A^2+B^2}}$$

to find α, divide the two equations

(*) to obtain

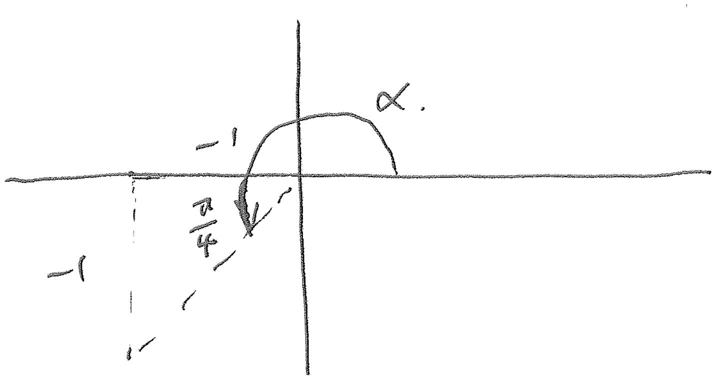
$$\frac{C\sin\alpha}{C\cos\alpha} = \frac{B}{A}.$$

$$\tan\alpha = \frac{B}{A}.$$

$$\text{“}\ \alpha = \tan^{-1}(B/A)\ \text{”}$$

be careful: the quantity B/A does not contain all the information required.

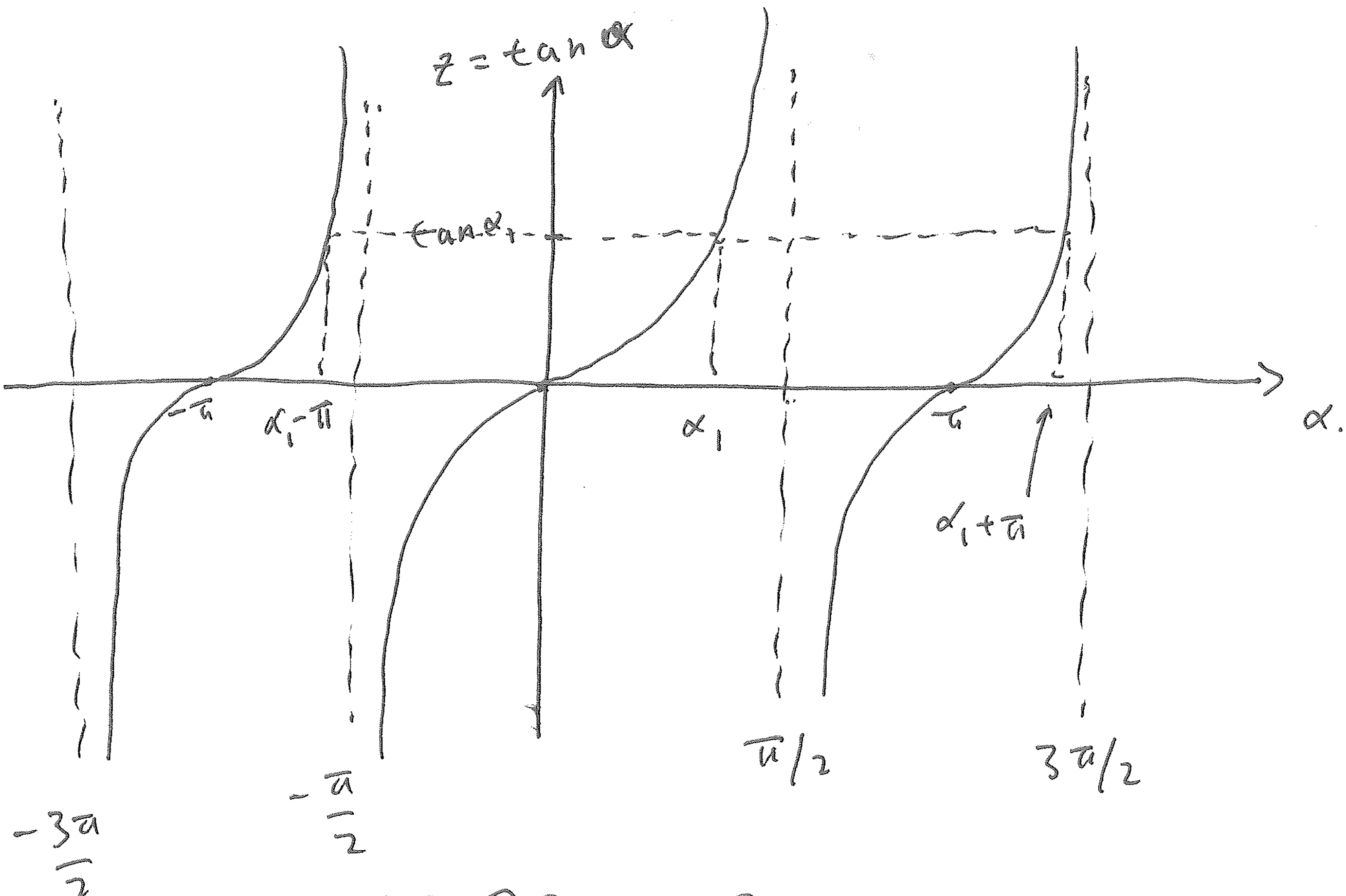
for example, if $A = -1$, $B = -1$.



then $\alpha = \pi + \frac{\pi}{4} = \frac{5\pi}{4}.$

however, <u>principal branch</u> of arctan (calculator or computer) gives

$$\tan^{-1}\left(\frac{-1}{-1}\right) = \tan^{-1}(1) = \frac{\pi}{4}.$$



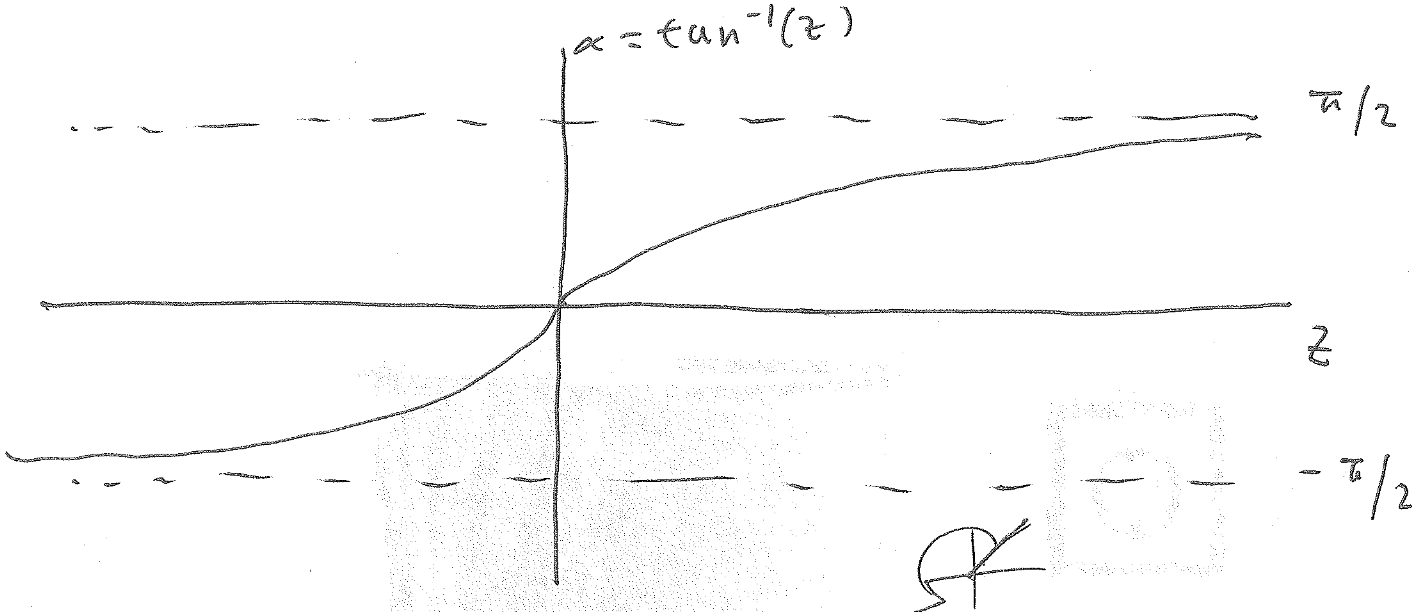
principal branch.

we can see that $\tan(\alpha + n\pi) = \tan\alpha$

for any integer n.

and so $\alpha = \tan^{-1}(z)$ is not unique

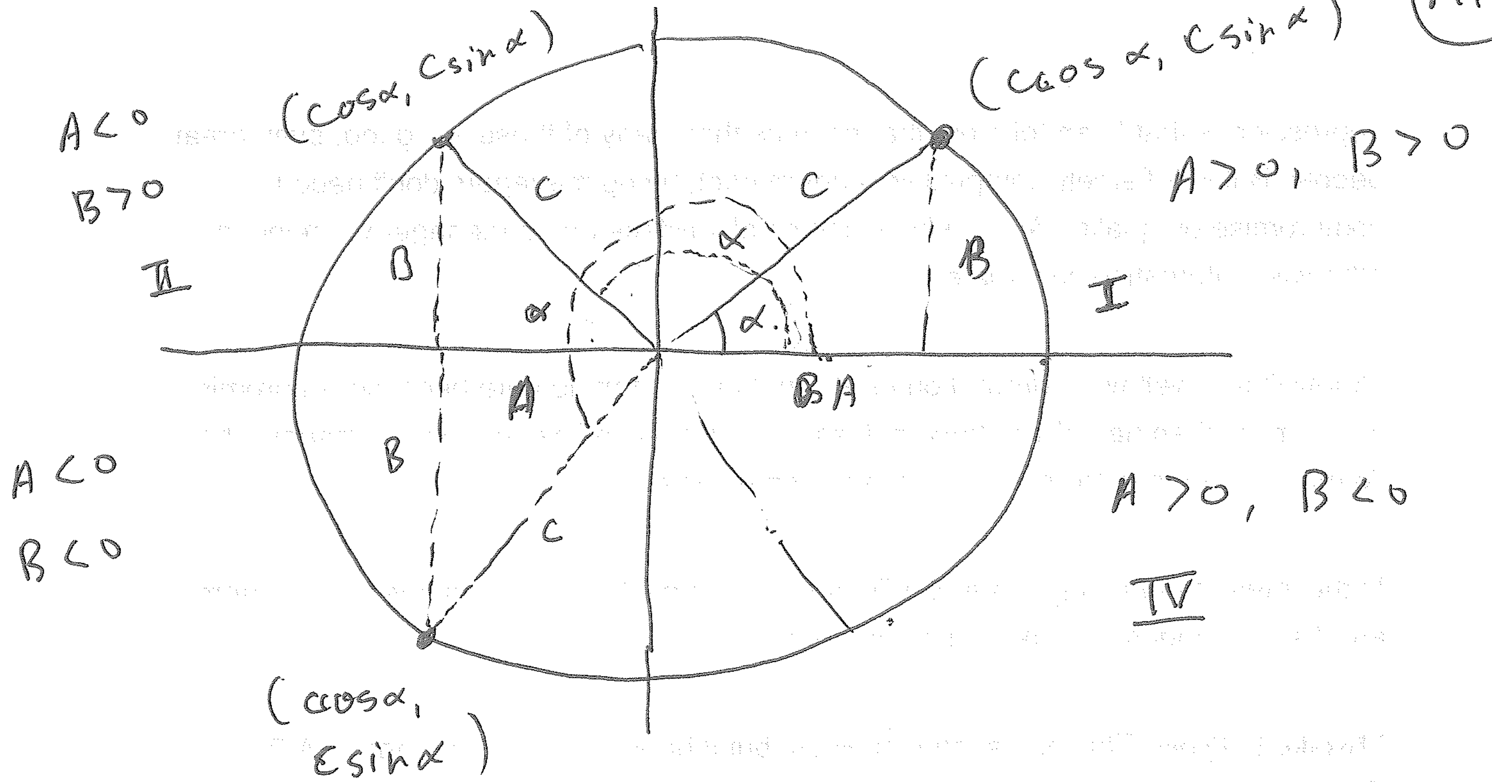
unless we restrict α to a certain branch. This branch $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$ is called the <u>principal branch</u>



that is

$$\boxed{-\frac{\pi}{2} < \tan^{-1}(z) < \frac{\pi}{2}}$$ important.

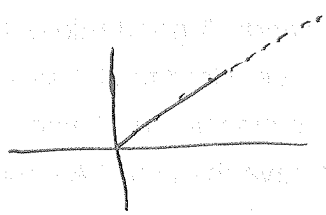
therefore, for $z = B/A$, which <u>does not</u> contain information about the signs of A and B <u>individually</u>, we need to add appropriate multiples of π to put α into the correct quadrant and so that $0 < \alpha < 2\pi$



$\tan^{-1}(z)$ gives you value in quadrants I or IV. may need to add π (or 2π) depending on signs of A and B.

↑ book

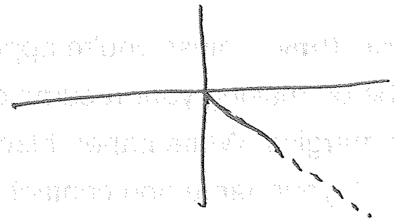
<u>Case 1</u>: <u>$A > 0$</u>



if $B > 0$

~~then $\tan^{-1}(B/A)$~~

then $0 < \tan^{-1}(B/A) < \frac{\pi}{2}$ → don't need to do anything if $B < 0$ (book)



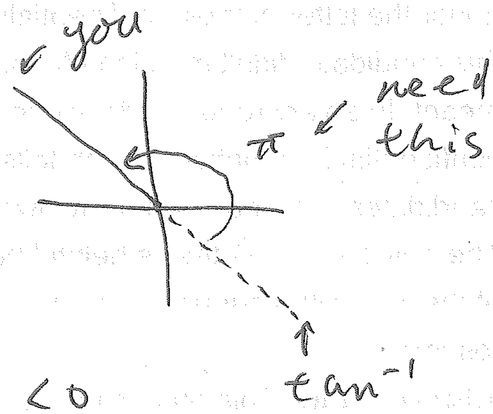
$-\frac{\pi}{2} < \alpha = \tan^{-1}(B/A) < 0$

(quadrant IV)

add 2π to α so that

$\frac{3\pi}{2} < \alpha < 2\pi$. (book requires $0 < \alpha < 2\pi$)

<u>case</u> $A < 0$

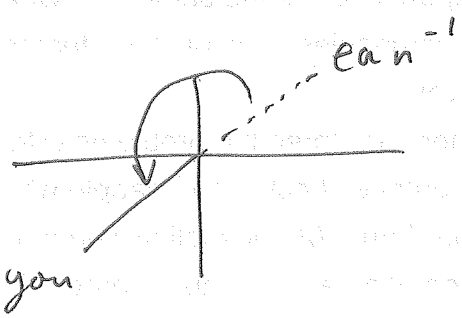


if $B > 0$,

then $-\frac{\pi}{2} < \alpha = \tan^{-1}(B/A) < 0$

need to add π to $\tan^{-1}(B/A)$

to put into second quadrant.



if $B < 0$

then $0 < \tan^{-1}(B/A) < \frac{\pi}{2}$.

need to add π ...

<u>summarize</u>

$$\alpha = \begin{cases} \left.\begin{array}{ll} \tan^{-1}(B/A) & \text{if } A>0,\ B>0 \\ 2\pi + \tan^{-1}(B/A) & \text{if } A>0,\ B<0 \end{array}\right\} A>0 \\ \tan^{-1}(B/A) + \pi \quad \text{if } A<0 \end{cases}$$

once we have C, α,

write

$$y = C\cos(\omega t - \alpha)$$

phase angle

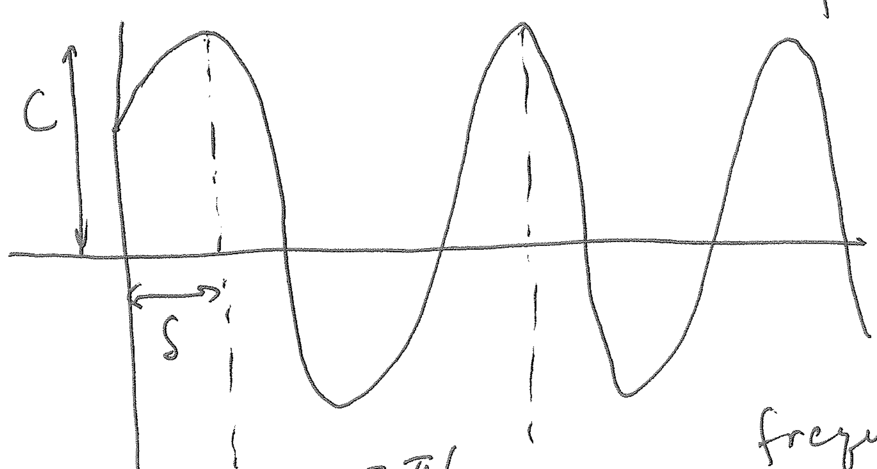
$$= C\cos(\omega(t - \alpha/\omega))$$

$$= C\cos(\omega(t - \delta)) \qquad \delta = \frac{\alpha}{\omega}$$

amplitude

circular frequency

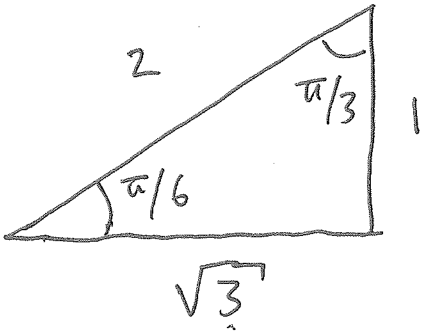
time lag (shifts cosine pattern to the right by δ)

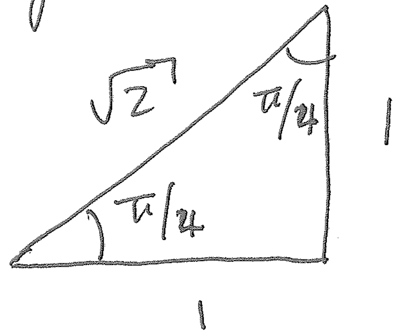


$T = 2\pi/\omega$ period.

frequency $\nu = \frac{1}{T} = \frac{\omega}{2\pi}$.

need to know two triangles.





EX find the solution to the IVP.

$$x'' + x = 0 \qquad x(0) = 1, \quad x'(0) = \sqrt{3}.$$

(write in phase-amplitude form)

$$x = A\cos t + B\sin t$$

$$x(0) = A = 1, \quad x'(0) = B = \sqrt{3}.$$

$$x = 1 \cdot \cos t + \sqrt{3}\sin t.$$

$r^2 = -1 \Rightarrow r = \pm i$

$e^{it} \quad e^{-it}$

$\uparrow$

$\cos t \pm i\sin t$

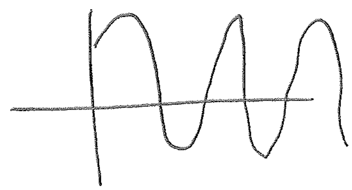
$(\omega = 1)$

$$C = \sqrt{1^2 + 3} = \sqrt{4} = 2.$$

$$\alpha = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$$

since $A > 0$, $B > 0$
do nothing.

$$\Rightarrow x = 2\cos(t - \pi/3).$$



Ex

write $\sqrt{3}\cos(3t) - \sin 3t = x(t)$

in p-a form.

$C = \sqrt{3+1} = 2.$

$\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = -\frac{\pi}{6}$

since $A > 0$, $B < 0$, add 2π so

$\alpha = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}.$

so $x = 2\cos\left(3t - \frac{11\pi}{6}\right)$

or

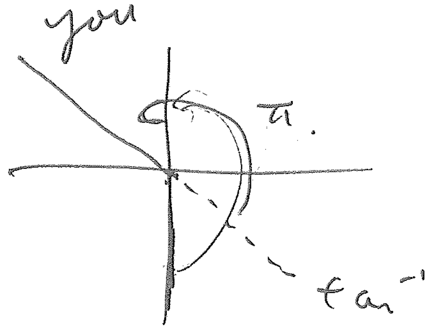
$x = 2\cos\left(3t + \frac{\pi}{6}\right)$

Ex write

$x = e^{-3t}(-\cos t + \sin t)$

in p-a form.

$c = \sqrt{(-1)^2 + 1^2} = \sqrt{2}.$



$\tan^{-1}\left(\frac{1}{-1}\right) = -\frac{\pi}{4}$

$\Rightarrow \alpha = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}.$

$x = \sqrt{2}\, e^{-3t} \cos\left(t - 3\pi/4\right).$

<u>EX</u>

write

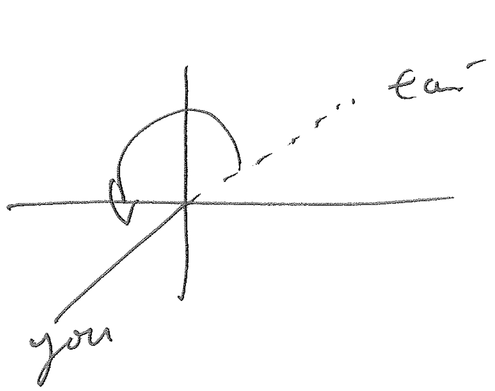
$x = -\sqrt{3}\cos 2t - \sin 2t$ in

p-a form

$c = \sqrt{3+1} = 2.$

$\tan^{-1}\left(\frac{-1}{-\sqrt{3}}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}.$

$\alpha = \frac{\pi}{6} + \pi = \frac{7\pi}{6}.$



$x = 2\cos\left(2t - \frac{7\pi}{6}\right).$

next time section 2.5.