Solution

MATH 2120 – Quiz 1 Thursday September 25, 2014

1. (a) Find the solution of the initial value problem

$$2\frac{dy}{dt} - y = e^{t/3}, \qquad y(0) = a.$$

(b) There exists a critical value a_0 for which $a < a_0$ and $a > a_0$ produce qualitatively different behaviors as $t \to \infty$. Determine the value of a_0 and the corresponding behaviors in the cases $a < a_0$ and $a > a_0$.

$$2\frac{dy}{dt} - y = e^{t/3} \qquad y(0) = a$$

put in standard form:

$$\frac{dy}{dt} - \frac{1}{2}y = \frac{1}{2}e^{t/3}$$

integrating factor $e^{\int -\frac{1}{2}\,dt} = e^{-t/2}$

$$\frac{d}{dt}\left(e^{-t/2}y\right) = \frac{1}{2}e^{t/3}e^{-t/2} = \frac{1}{2}e^{-t/6}$$

$$e^{-t/2}y = -\frac{1}{2}\cdot 6e^{-t/6} + C$$

$$y = -3e^{t/3} + Ce^{t/2}$$

$y(0) = a \Rightarrow -3 + C = a \Rightarrow C = a+3$

$$\boxed{y = -3e^{t/3} + (a+3)e^{t/2}} = e^{t/2}\left[-3e^{-t/6} + a + 3\right]$$

as $t \to \infty$, $y \sim (a+3)e^{t/2}$ so $\boxed{a_0 = -3}$

$$\boxed{a > -3 \Rightarrow y \to \infty, \quad a < -3 \Rightarrow y \to -\infty}$$

2. Solve the initial value problem

$$y'(x) = 4\frac{xy^3}{\sqrt{1+2x^2}}, \qquad y(0) = 1,$$

and determine the interval on which the solution exists.

$$\frac{dy}{dx} = \frac{4xy^3}{\sqrt{1+2x^2}} \qquad \text{separable :}$$

$$\int y^{-3}\,dy = \int \frac{4x}{\sqrt{1+2x^2}}\,dx$$

$$\frac{y^{-2}}{-2} = 2\sqrt{1+2x^2} + c.$$

$$\frac{1}{y^2} = -4\sqrt{1+2x^2} + c$$

$$y = \frac{1}{\sqrt{c - 4\sqrt{1+2x^2}}}$$

$$y(0) = \frac{1}{\sqrt{c-4}} = 1 \Rightarrow c = 5$$

$$\boxed{y = \frac{1}{\sqrt{5 - 4\sqrt{1+2x^2}}}}$$

existence requires $5 - 4\sqrt{1+2x^2} > 0$

$$\frac{5}{4} > \sqrt{1+2x^2} \Rightarrow \frac{25}{16} - 1 > 2x^2$$

$$\frac{9}{32} > x^2 \Rightarrow \boxed{|x| < \frac{3}{\sqrt{32}}}$$

interval of existence.