Solutions

MATH 2120 – Quiz 4   Thursday October 23, 2014

1. Find the particular solution of the equation

$$y'' + 4y = 2\sin^2 x\,.$$

2. Write down the form of the particular solution of the equation

$$y'' - 7y' + 10y = x^3 e^{5x} + x^2\,.$$

Do not solve for the coefficients.

3. Variation of parameters states that the particular solution of $y'' + p(t)\,y' + q(t)\,y = f(t)$ is given by

$$y_p(t) = y_1(t)\int \frac{-y_2(t)f(t)}{W(y_1,y_2\,;t)}\,dt + y_2(t)\int \frac{y_1(t)f(t)}{W(y_1,y_2\,;t)}\,dt\,, \qquad (1)$$

where $W(y_1, y_2\,; t)$ is the Wronskian of the two linearly independent homogeneous solutions $y_1(t)$ and $y_2(t)$. For the equation

$$y'' + 9y = \cos 3t\,,$$

set up the two integrals in (1). Choose **one** to evaluate.

(1) $y'' + 4y = 2\sin^2 x = 1 - \cos 2x$

$y_1 = \sin 2x, \quad y_2 = \cos 2x$

$y_p = y_{p_1} + y_{p_2}$ where $y_{p_1}'' + 4y_{p_1} = 1 \Rightarrow y_{p_1} = \frac{1}{4}$

$y_{p_2} = x\left[A\cos 2x + B\sin 2x\right]$

$y_{p_2}'' = x\left[-4A\cos 2x - 4B\sin 2x\right] + 2\left[-2A\sin 2x + 2B\cos 2x\right]$

$\Rightarrow 2\left[-2A\sin 2x + 2B\cos 2x\right] = -\cos 2x$

$\Rightarrow 4B = -1, \quad A = 0 \Rightarrow B = -\frac{1}{4}$

so $\boxed{y_p = \frac{1}{4} - \frac{1}{4}x\sin 2x}$

(2) $y'' - 7y' + 10y = x^3 e^{5x} + x^2$

C.E : $(r-5)(r-2) = 0 \Rightarrow r_1 = 5,\ r_2 = 2$

so $y_1 = e^{5x}, \quad y_2 = e^{2x}$

then we must have

$y_p = y_{p_1} + y_{p_2}$

$y_{p_1} = Ax^2 + Bx + C$

$y_{p_2} = x[Dx^3 + Ex^2 + Fx + G]e^{5x}$

$$\Rightarrow \boxed{y_p = Ax^2 + Bx + C + x\left[Dx^3 + Ex^2 + Fx + G\right]e^{5x}}$$

(3) $y'' + 9y = \cos 3t \qquad y_1 = \cos 3t, \quad y_2 = \sin 3t$

$$W = \begin{vmatrix} \cos 3t & \sin 3t \\ -3\sin 3t & 3\cos 3t \end{vmatrix} = 3\left[\cos^2 3t + \sin^2 3t\right] = 3$$

$$\left.\begin{aligned} &\Rightarrow \int \frac{-y_2 f}{W}\,dt = -\frac{1}{3}\int \sin 3t \cos 3t\,dt \\ &\int \frac{y_1 f}{W}\,dt = \frac{1}{3}\int \cos^2 3t\,dt \end{aligned}\right\} f(t) = \cos 3t$$

now

$$-\frac{1}{3}\int \sin 3t \cos 3t\,dt = -\frac{1}{3}\int \frac{1}{6}\frac{d}{dt}\sin^2 3t\,dt = -\frac{1}{18}\sin^2 3t$$

or $-\frac{1}{3}\int \sin 3t\cos 3t\,dt = \frac{-1}{3}\int \frac{1}{2}\sin 6t\,dt = \frac{1}{36}\cos 6t$

$C = 0$ wlog

$$\frac{1}{3}\int \cos^2 3t\,dt = \frac{1}{3}\int \frac{1+\cos 6t}{2} = \frac{t}{6} + \frac{1}{36}\sin 6t$$

$C = 0$ wlog.