Solution

MATH 2120 – Quiz 5 Thursday November 13, 2014

1. Solve for $\boxed{x(t)}$ and $y(t)$ in the following system of equations using Laplace transforms. Note the different initial conditions from the question in the homework.

Only had to solve for $x(t)$

$$x' = 2x + y\,; \qquad x(0) = 6\,,$$
$$y' = 6x + 3y\,; \qquad y(0) = -2\,.$$

2. Consider the Volterra integral equation

$$\phi(t) + \int_0^t (t - \xi)\,\phi(\xi)\,d\xi = \sin(2t)\,. \tag{1}$$

Solve (1) by using the Laplace transform.

$x' = 2x + y \qquad x(0) = 6$

$y' = 6x + 3y \qquad y(0) = -2$

$$\left.\begin{aligned} sX(s) - 6 &= 2X + Y \\ sY + 2 &= 6X + 3Y \end{aligned}\right\} \quad \begin{aligned} (s-2)X - Y &= 6 \\ -6X + (s-3)Y &= -2 \end{aligned}$$

$$\begin{pmatrix} s-2 & -1 \\ -6 & s-3 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}.$$

$$X = \frac{\begin{vmatrix} 6 & -1 \\ -2 & s-3 \end{vmatrix}}{(s-2)(s-3) - 6} = \frac{6(s-3) - 2}{s^2 - 5s + 6 - 6}$$

$$= \frac{6s - 20}{s(s-5)} = \frac{A}{s} + \frac{B}{s-5}$$

$6s - 20 = A(s-5) + Bs$

$\underline{s=5}$: $10 = 5B \Rightarrow B = 2!$

$\underline{s=0}$: $-20 = -5A \Rightarrow A = 4$

$\Rightarrow X = \frac{4}{s} + \frac{2}{s-5}.$

$x(t) = 4 + 2e^{+5t}.$

$$\phi + \int_0^t (t-\xi)\,\phi(\xi)\,d\xi = \sin 2t.$$

$$\phi + \frac{1}{s^2}\Phi = \frac{2}{s^2+4}.$$

$$\frac{s^2\Phi + \Phi}{s^2} = \frac{2}{s^2+4}.$$

$$\Phi = \frac{2s^2}{(s^2+4)(s^2+1)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+1}$$

$$2s^2 = (As+B)(s^2+1) + (Cs+D)(s^2+4)$$

$s=i$ $\quad -2 = (iC+D)(+3)$

$$\Rightarrow C=0,\quad D=-\frac{2}{3}.$$

$s=2i$: $\quad -8 = (A\cdot 2i+B)(-3)$

$$\Rightarrow A=0,\quad B=\frac{8}{3}$$

$$\Phi = \frac{8}{3}\frac{1}{s^2+4} - \frac{2}{3}\frac{1}{s^2+1}$$

$$= \frac{4}{3}\frac{2}{s^2+4} - \frac{2}{3}\frac{1}{s^2+1}.$$

$$\boxed{\phi = \frac{4}{3}\sin 2t - \frac{2}{3}\sin t.}$$