Taylor Series Review

- a Taylor series of a function $f(x)$ centered at point $x=a$ is an infinite sum of non-negative powers of $(x-a)$:

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \qquad (\ast)$$

assuming f is infinitely differentiable at $x = a$

- here, $n! = n(n-1)(n-2)\dots(2)(1)$
  $0! = 1$

- notice: global information of $f(x)$ can be obtained from only knowing its derivatives at one point

- if $a = 0$, the series is referred as a Maclaurin series

- the ~~rd~~ "radius of convergence" of $\ast$ depends on the location of the point a in relation to singularities of f (perhaps in the complex plane)

- for example, $f(x) = \frac{1}{1-x}$ is singular at $x=1$, so its Maclaurin series

$$\frac{1}{1-x} = 1 + x + x^2 + \dots$$

is valid only on the interval

$$-1 < x < 1$$

- the same applies for the Maclaurin series for

$f(z) = \frac{1}{1+z^2}$ even though it has no singularities on the real line (singularities are at $z = \pm i$)

- how do we derive the coeff's in A?

- assume that $f(x)$ can be written as a series

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots$$

T5

to compute c_0, set $x=a$,

$$f(a) = c_0 + 0 + 0 + \dots$$

$$\Rightarrow c_0 = f(a)$$

to compute c_1, differentiate once w.r.t. x

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \dots$$

then set $x=a$,

$$f'(a) = c_1 + 2\cdot 0 + 0 \dots$$

$$\Rightarrow c_1 = f'(a)$$

c_2:

$$f''(x) = 2c_2 + 3\cdot 2c_3(x-a) + 4\cdot 3c_4(x-a)^2 + \dots$$

set $x = a$

$$f''(a) = 2c_2 \Rightarrow c_2 = \frac{1}{2}f''(a)$$

⋮

etc

- note that when x is very close to a, that is $|x-a| \underbrace{\ll}_{\text{much less than}} 1$, the first few terms of the Taylor series gives a very good approx. for $f(x)$

Since $(x-a) >> (x-a)^2 >> (x-a)^3 \ldots$

- so let $\underbrace{x = a+h \quad |h| << 1}_{\text{says } x \text{ close to } a,}$

then

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2} f''(a) \ldots$$

[same form as we used in class

$$u(x+\Delta x) = u(x) + \Delta x\, u'(x) + \frac{\Delta x^2}{2} u''(x) \ldots]$$

- some common Taylor (Maclaurin) series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots \quad -1 < x < +1$$

$$(1-x)^{1/2} = 1 - \frac{1}{2}x - \frac{x^2}{8} - \ldots \quad -1 < x < 1$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \ldots$$

(note: all odd powers)

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

(note only even powers)

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

$$\left[\frac{d}{dx} e^x = e^x\right]$$

$$\sinh x = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots$$

→ to be used later

$$\cosh x = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$$

- an other important identity to be used later:

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} + \ldots$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\right)$$

$= \cos x + i \sin x$

$$\left[\ e^{i\pi} = -1 \rightarrow \underline{\underline{e^{i\pi} + 1 = 0}}\ \right]$$

- why are Taylor series important?
  - derive diff. eqns from discrete ~~rp~~ representation
  - linearize or simplify diff. eqns

    eg. $\dfrac{d^2\theta}{dt^2} = -\sin\theta \simeq \underline{\theta}$
  - integrability of a function (use later in Laplace Transforms)

eg. is $f(x) = \dfrac{1-\cos x}{\sin x}$ integrable at $x = 0$?

near $x = 0$,

$$1 - \cos x \sim 1 - \left(1 - \frac{x^2}{2} + O(x^4)\right)$$

$$= \frac{x^2}{2} + \underset{\uparrow \text{ order}}{O}(x^4)$$

$$\sin x \sim x + O(x^3)$$

$$\Rightarrow \frac{1-\cos x}{\sin x} \sim \frac{x^2/2}{x} = \frac{1}{2}x \text{ as}$$

$x \to 0$

$\Rightarrow f(x)$ is integrable near $x = 0$

what about

$$g(x) = \frac{\cos x}{\sin x} e^{-x}?$$

$\cos x \sim 1 + O(x^2)$

$e^{-x} \sim 1 + O(x)$

$\sin x \sim x + O(x^3)$

$\Rightarrow g(x) \sim \frac{1}{x}$ as $x \to 0$

so not integrable

- linear stability analysis (3120)

- remark: a function can be infinitely diff'able at a point but not be equal to its Taylor series around that point

e.g. $f(x) = e^{-1/x^2}$

because $e^{-1/x^2} \to 0$ rapidly as $x \to 0$ (faster than any polynomial) all derivatives of f at $x=0$ are identically 0. Therefore, its Maclaurin series is 0, even though f is nonzero for $x \neq 0$