# Homogeneous first order equations (1.6)

- note: "homogeneous" has a very different meaning in other context, in particular for higher order equations
- first order homogeneous eqns have the form

$$\frac{dy}{dx} = F(y/x) \qquad (*)$$

that is, $f(x,y)$ can be written as a function of

~~y/x~~ only $y/x$

note: $f(rx, ry) = f(x, y)$

for any constant $r$

because $f(rx, ry) = F(ry/rx)$

$= F(y/x) = f(x, y)$

to solve, pose substitution

$$v(x) = \frac{y(x)}{x} \rightarrow y = xv$$

and substitute into $*$:

$$\frac{dy}{dx} = \frac{d}{dx}(xv) = v + x\frac{dv}{dx}$$

and

$$F(y/x) = F(v)$$

then (*) becomes

$$v + x\frac{dv}{dx} = F(v)$$

$$x\frac{dv}{dx} = F(v) - v$$

$$\left.\frac{dv}{dx} = \frac{F(v) - v}{x}\quad\right\}\text{separable}$$

↳ $$\int \frac{dv}{F(v) - v} = \int \frac{dx}{x} \qquad (+)$$

solve (+) for $v(x)$, then

compute $y(x) = x\,v(x)$

Ex find the general soln of

$$\frac{dy}{dx} = \frac{2xy}{x^2 - y^2}$$

and leave in implicit form

check if homogeneous:

$x \to rx, \quad y \to ry.$

$$\frac{2xy}{x^2-y^2} \longrightarrow \frac{2rx \cdot ry}{(rx)^2 - (ry)^2}$$

$$= \frac{2r^2xy}{r^2(x^2-y^2)} = \frac{2xy}{x^2-y^2}$$

therefore, homogeneous.

divide numerator and denominator by $\frac{1}{xy}$

$$\frac{dy}{dx} = \frac{2}{\frac{x^2}{xy} - \frac{y^2}{xy}} = \frac{2}{\frac{x}{y} - \frac{y}{x}}$$

$1/v$ $v$

so we have the form

$$\frac{dy}{dx} = F(y/x)$$

let $v = y/x$, then we have

$$x\frac{dv}{dx} = F(v) - v = \frac{2}{\frac{1}{v} - v} - v$$

$$= \frac{v(v^2+1)}{1-v^2}$$

(H6)

$$x\frac{dv}{dx} = \frac{v(v^2+1)}{1-v^2}$$

separate variables:

$$\int \frac{1-v^2}{v(v^2+1)}\,dv = \int \frac{1}{x}\,dx$$

use partial fractions to compute LHS:

$$\frac{1-v^2}{v(\underbrace{v^2+1})} = \frac{1-v^2}{v(v+i)(v-i)}$$

$$= \frac{A}{v} + \frac{B}{v+i} + \frac{C}{v-i}$$

$$\left[\,(v+i)(v-i) = v^2+iv-iv+1 = v^2+1\,\right]$$

$$\frac{1-v^2}{v(v^2+1)} = \frac{A(v+i)(v-i) + Bv(v-i) + Cv(v+i)}{v(v+i)(v-i)}$$

set :

$v = i$ : $1-(i)^2 = C(i)(2i)$

$= -2C$

$2 = -2C \Rightarrow C = -1$

$v = -i$ :

$1-(-i)^2 = 2 = B(-i)(-2i)$

$= -2B$

$\Rightarrow B = -1$

$v = 0$ :

$1 = A(i)(-i) = A$

$\Rightarrow A = 1$

therefore

$$\frac{1-v^2}{v(v^2+1)} = \frac{1}{v} - \frac{1}{v+i} - \frac{1}{v-i}$$

$$= \frac{1}{v} - \left[\frac{1}{v+i} + \frac{1}{v-i}\right]$$

$$= \frac{1}{v} - \frac{2v}{v^2+1}$$

then

$$\int \frac{1-v^2}{v(v^2+1)}\,dv = \log v - \log(v^2+1)$$

therefore,

$$\log \frac{v}{v^2+1} = \log x + c$$

$$= \log(\tilde{c}\,x)$$

$$(c = \log \tilde{c})$$

then

$$\frac{v}{v^2+1} = \tilde{c}\, x$$

finally, $y = xv$ or $v = y/x$

$$\frac{y/x}{y^2/x^2+1} = \tilde{c}\, x$$

fine ... implicit
can solve explicitly for
$y(x)$ using quadratic
formula

Ex

solve $\frac{dy}{dx} = \frac{y+x}{x} = \frac{y}{x} + 1$

"v"

(this is also a linear equation so can solve using integrating factor)

let $v = y/x$, then

$$\cancel{v} + x\frac{dv}{dx} = \cancel{v} + 1$$

$$\frac{dv}{dx} = \frac{1}{x} \quad \text{(separable)}$$

$$\int dv = \int \frac{1}{x}\, dx$$

$$v = \log|x| + C$$

log

Log

now $y = xv$

$$y(x) = x \log x + cx$$

<u>Ex</u>

$$\frac{dy}{dx} = \frac{xy + y^2 + x^2}{y^2}$$

divide num. and denom by $x^2$:

$$\frac{dy}{dx} = \frac{\frac{xy}{x^2} + \frac{y^2}{x^2} + 1}{(y/x)^2}$$

$$= \frac{y/x + (y/x)^2 + 1}{(y/x)^2}$$

with $v = y/x$

RHS becomes $\dfrac{v + v^2 + 1}{v^2}$

then solve

$$v + x\frac{dv}{dx} = \frac{v + v^2 + 1}{v^2}$$

(separable)

next time $\underline{\underline{2.1}}$