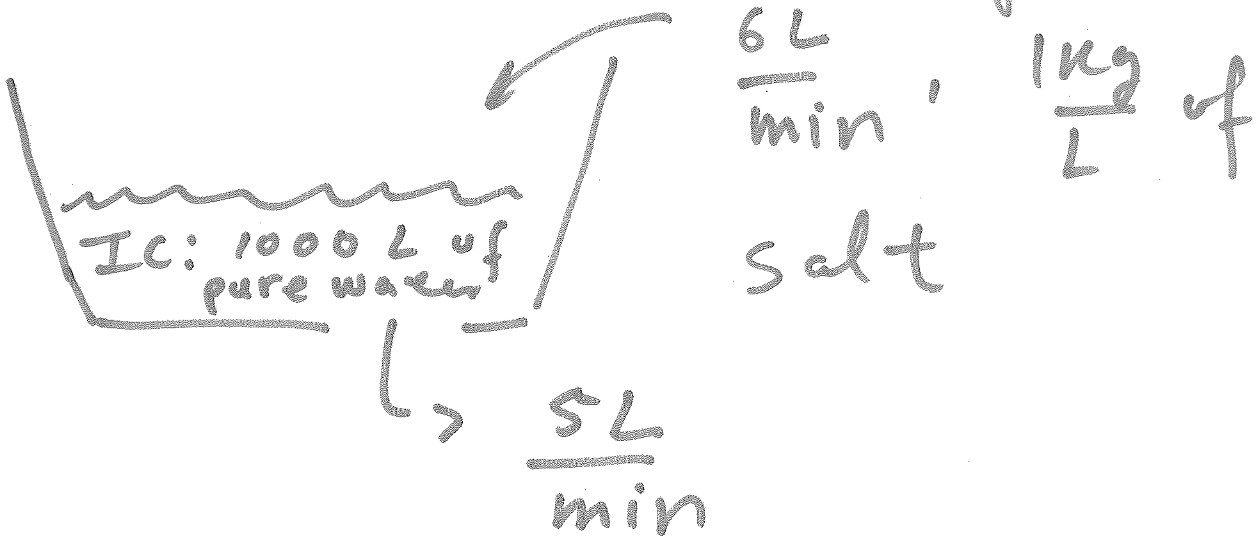
continuation of linear equations



find concentration of salt $c(t)$ at time t (assume well-mixed)

let $x(t)$ be mass of salt in tank at time t

let $V(t)$ be total volume of solution. Then $c(t) = \dfrac{x(t)}{V(t)}$

(L2)

differential equation for $V(t)$:

$$\frac{dV}{dt} = \underbrace{\frac{6\,L}{min}}_{\text{input}} - \underbrace{\frac{5\,L}{min}}_{\text{output}} \quad ; \; V(0)=0$$

$$\frac{dV}{dt} = 1 \Rightarrow V = t + C$$

$$V(0) = 1000 \Rightarrow V = t + 1000$$

diff. eqn for $x(t)$:

$$\frac{dx}{dt} = \underbrace{\frac{6L}{min} \cdot 1\frac{kg}{L}}_{\text{mass input}} - \underbrace{\frac{5L}{min} \overbrace{\boxed{\frac{x(t)}{V(t)}}}^{\text{current concentration}}}_{\text{mass output}}$$

$$= 6 - \frac{5\,x}{1000+t} \qquad x(0) = 0$$

then

$$\frac{dx}{dt} + \underbrace{\frac{5}{1000+t}}_{\text{"}p(t)\text{"}}x = \underbrace{6}_{\text{"}q(t)\text{"}} \qquad (*)$$

integrating factor

$$u(t) = e^{\int p(t)\,dt}$$

$$= e^{5\log(1000+t)}$$

$$= \underline{(1000+t)^5} \qquad y(t) = \frac{\quad}{u(t)}$$

~~then we~~ multiply everything in $(*)$ by $u(t)$ to obtain

$$\frac{d}{dt}\left((1000+t)^5 x(t)\right) = 6(1000+t)^5$$

$$(1000+t)^5 x(t) = (1000+t)^6 + C$$

so

$$x(t) = 1000 + t + \frac{c}{(1000+t)^5}$$

$x(0) = 0$

$$\Rightarrow 1000 + 0 + \frac{c}{1000^5} = 0$$

$$\Rightarrow c = -1000^6$$

$$x(t) = 1000 + t - \frac{1000^6}{(1000+t)^5}$$

concentration $c(t)$ is

$$c(t) = \frac{x(t)}{V(t)} = 1 - \frac{1000^6}{(1000+t)^6}$$

as $t \to \infty$, $c \to 1^-$

(this is the concentration of the input solution)

Bernoulli Equations (1.6)

- have the form

$$\frac{dy}{dx} + p(x)\,y = q(x)\,y^n \quad (+)$$

mistake (boo boo)

[$n = 0 \to$ linear

$n = 1 \to$ separable and linear]

- can turn (+) into a linear equation by using the substitution

$$v = y^{1-n},$$

$\to$ get linear eqn for $v(x)$

$$v = y^{1-n}$$

$$y = v^{\frac{1}{1-n}}$$

$$\frac{dy}{dx} = \frac{1}{1-n} \, v^{\frac{1}{1-n} - 1} \, \frac{dv}{dx}$$

$$\frac{1}{1-n} - 1 = \frac{1}{1-n} - \frac{1-n}{1-n} = \frac{n}{1-n}$$

$$\frac{dy}{dx} = \frac{1}{1-n} \, v^{\frac{n}{1-n}} \, \frac{dv}{dx}$$

put in (+)

$$\frac{1}{1-n} \, v^{\frac{n}{1-n}} \, \frac{dv}{dx} + p(x) \, v^{\frac{1}{1-n}}$$

$$= q(x) \, v^{\frac{n}{1-n}}$$

$$\frac{1}{1-n}\frac{dv}{dx} + p(x)\, v^{\frac{1}{1-n} - \frac{n}{1-n}}$$

$$= q(x)\, v^{\frac{n}{1-n} - \frac{n}{1-n}}$$

$$\frac{1}{1-n}\frac{dv}{dx} + p(x)\, v = q(x)$$

$$\frac{dv}{dx} + (1-n)\, p(x)\, v = (1-n)\, q(x)$$

procedure: solve for v using integrating factor then find y by

$$y(x) = v^{\frac{1}{1-n}}$$