D1

Second order linear equations

(2.1)

- a second order linear equation has the form

$$(1) \quad \frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)\,y = f(t)$$

- observe that (1) is linear in y, y', y''

- notation: if we define a linear operator L as

$$L = \frac{d^2}{dt^2} + p(t)\frac{d}{dt} + q(t),$$

then (1) can be written as

$$L(y) = f(t)$$

- if $f(t) \equiv 0$ (indentically 0 for all t), we say that (1) is a <u>homogeneous</u> equation. otherwise (1) is <u>nonhomogeneous</u>. For example,

$$L(y) = e^t$$

is non-homogeneous, and $L(y)=0$ is the

associated homogeneous equation

- <u>any</u> linear operator has the property

$$L(c_1 y_1 + c_2 y_2) = c_1 L(y_1) + c_2 L(y_2)$$

- for L in (2) (example),

$$L(c_1 y_1 + c_2 y_2)$$

$$= \frac{d^2}{dt^2}(c_1 y_1 + c_2 y_2) + p(t)\frac{d}{dt}(c_1 y_1 + c_2 y_2)$$

$$+ q(t)(c_1 y_1 + c_2 y_2)$$

$$= c_1 \frac{d^2}{dt^2} y_1 + c_1 p(t) \frac{d}{dt} y_1 + c_1 q(t) y_1$$

$$+ c_2 \frac{d^2}{dt^2} y_2 + c_2 p(t) \frac{d}{dt} y_2 + c_2 q(t) y_2$$

D4

$$= c_1 \left[y_1'' + p(t)\, y_1' + q(t)\, y \right]$$

$$+ c_2 [\ \dots\ ]$$

$$= c_1 L(y_1) + c_2 L(y_2)$$

- this leads to the principle of superposition for homogeneous equations:

let $y_1(t)$ and $y_2(t)$ be solutions of the homog. eqn $L(y) = 0$ [ie, $L(y_1) = L(y_2) = 0$]. Then $y = c_1 y_1 + c_2 y_2$ is also a soln

proof: $L(y) = L(c_1 y_1 + c_2 y_2)$

$$= c_1 L(y_1) + c_2 L(y_2) = c_1 \cdot 0 + c_2 \cdot 0$$

$$= 0$$

<u>EX</u> $y_1 = \cos mt$ and $y_2 = \sin mt$

are solutions of the second order ODE

$$y'' = -m^2 y \qquad (3)$$

verify: $\frac{d}{dt} \cos mt = -m \sin mt$.

$$\frac{d^2}{dt^2} \cos mt = -m^2 \cos mt$$

(that is, 2 deriv's of $\cos mt$ gives $-m^2$ times itself)

then

$$y = c_1 \cos mt + c_2 \sin mt$$

for <u>any</u> c_1, c_2 is a solution of (3).

- notice also that

$$y_1 = e^{imt} \quad \text{and} \quad y_2 = e^{-imt}$$

also solve (3)

$$y_1' = im\, e^{imt} \qquad y_1'' = (im)(im)\, e^{imt}$$
$$= -m^2 e^{imt}$$
$$= -m^2 y_1$$

then we have

$$c_1 y_1 + c_2 y_2 = c_1 (\cos mt + i \sin mt)$$
$$+ c_2 (\cos mt - i \sin mt)$$

$$= (c_1 + c_2) \cos mt + i (c_1 - c_2) \sin mt.$$

let $A = c_1 + c_2$, $\quad B = i(c_1 - c_2)$

$$= A \cos mt + B \sin mt$$

to fix A, B, require $y(a)$ and and $y'(a)$ to be specified note: no condition on y''

<u>Ex</u> let $y(0) = 1$, $y'(0) = 2$

then

$$y(0) = 1 = A \Rightarrow A = 1$$

$$y'(0) = 2 = mB \Rightarrow B = \frac{2}{m}.$$

then $y(t) = 1 \cdot \cos mt + \frac{2}{m} \sin mt.$

claim: <u>this is then only soln</u>

<u>Existence and uniqueness for second order linear equations:</u>

consider the IVP

$$y'' + p(t) y' + q(t) y = f(t) \quad (4)$$

where $y(a) = b_0$, $y'(a) = b_1$

where p, q, f are continuous on some interval I $\alpha < t < \beta$ that contains a. Then (4) has one and only one solution on I.

↑ existence ↑ uniqueness

proof: simple extension of proof for first order ODE's (posted on website)

Ex on what interval does a unique solution to the IVP below exist?

$$y'' + \frac{1}{t} y' + t^2 y = 0 , \quad y(1) = 0, \quad y'(1) = 1$$

$p(t) = \frac{1}{t}$ continuous on $-\infty < t < 0$, $0 < t < \infty$.

$q(t) = t^2$ cont's everywhere,

$f(t) = 0$ " "

~~soln~~ so soln exists and is unique on $0 < t < \infty$

Linear independene

let $\quad L \equiv \frac{d^2}{dt^2} + p(t)\frac{d}{dt} + q(t)$

suppose $\quad L(y_1) = 0 \quad \Big| \quad p, q$ cont's
$\quad\quad\quad\quad L(y_2) = 0 \quad \Big| \quad$ on interval I.

can we express the soln of the IVP

$$L(y) = 0, \quad y(a) = b_0, \quad y'(a) = b_1$$

as a linear combination of y_1 and y_2

for <u>any</u> a, b_0, b_1 ?

answer: only if y_1 and y_2 are linearly independent (l.i)

- two functions $f(t)$ and $g(t)$ are ~~linearly~~ l.i if there exist

c_1 and c_2 <u>not both 0</u> s.t.

$$c_1 f + c_2 g = 0$$

for all t.

(that is, f is not a constant multiple of g)

<u>Ex</u>

$f(x) = x$, $g(x) = 3x$ are linearly dependent (l.d.) since

$$\frac{g(x)}{f(x)} = 3 \text{ is a constant.}$$

$f(x) = \sin x$, $g(x) = \cos x$, are l.i.

since $\dfrac{g(x)}{f(x)} = \cot x \neq$ constant.

<u>Wronskian of solutions</u>

suppose $L(y_1) = L(y_2) = 0$

and consider the Wronskian

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

$W(y_1, y_2; x)$ determinant

- if y_1, y_2 are l.d. on I, then $W \equiv 0$ on I (notice if $y_1 = k y_2$, then

$$W = k y_2 y_2' - y_2 \cdot k y_2' = 0$$

- if y_1, y_2 are l.i on I, $W \neq 0$ <u>everywhere</u> on I (show later)

i.e. W is either <u>always</u> 0, or <u>never</u> 0

motivation for definition of W:

try to solve IVP

$L(y) = 0, \quad y(a) = b_0, \quad y'(a) = b_1$

to see if

$$y = c_1 y_1 + c_2 y_2$$

is a viable solution, try to solve for c_1, c_2 ($L(y_1) = L(y_2) = 0$)

if we can solve for c_1 and c_2, then solution of (*) may be expressed as a linear combination of y_1 and y_2. Otherwise it cannot.

IC's:

$$y(a) = b_0 \Rightarrow c_1 y_1(a) + c_2 y_2(a)$$

cont'd next time.