Wronskians of solutions

$$L \equiv \frac{d^2}{dt^2} + p(t)\frac{d}{dt} + q(t)$$

p, q cont's on interval I

suppose $L(y_1) = L(y_2) = 0$

and consider the Wronskian

$$W(y_1, y_2) \equiv \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

- if y_1, y_2 are l.d. on I, then

  $W \equiv 0$ on I

- if y_1, y_2 are l.i. on I, then

  $W \neq 0$ everywhere on I (show later)

i.e., W is either always 0 or never 0

motivation for definition of W:

try to solve IVP

$$L(y)=0, \quad y(a)=b_0, \quad y'(a)=b_1 \quad (*)$$

to see if $y = c_1 y_1 + c_2 y_2$ is a viable solution, $(L(y_1) = L(y_2) = 0)$

try to solve for c_1, c_2. If we can solve for c_1, c_2, the solution of $(*)$ ~~mg~~ may be expressed as a linear combination of y_1, y_2. otherwise it cannot.

solve for c_1, c_2:

$$y(a) = b_0 \Rightarrow c_1 y_1(a) + c_2 y_2(a) = b_0$$

$$y'(a) = b_1 \Rightarrow c_1 y_1'(a) + c_2 y_2'(a) = b_1$$

write in matrix form:

$$\begin{pmatrix} y_1(a) & y_2(a) \\ y_1'(a) & y_2'(a) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \end{pmatrix}$$

for solution ~~of~~ for c_1, c_2 to exist, require the determinant of the matrix (which we call the wronskian) to be nonzero.

[to solve for c_1:

$$c_1 y_1(a) y_2'(a) + \cancel{c_2 y_2(a) y_2'(a)} = y_2'(a) b_0$$

$$- \quad c_1 y_1'(a) y_2(a) + \cancel{c_2 y_2'(a) y_2(a)} = y_2(a) b_1$$

$$c_1 [y_1(a) y_2'(a) - y_1'(a) y_2(a)] = y_2'(a) b_0 - y_2(a) b_1$$

$$c_1 = \frac{y_2'(a)\, b_0 - y_2(a)\, b_1}{y_1(a) y_2'(a) - y_1'(a) y_2(a)}$$

denominator is the wronskian. if nonzero, soln for c_1 exists

Ex $y_1 = e^{2x}$, $y_2 = e^{-2x}$ are soln's of $y'' - 4y = 0$

show $w(y_1, y_2) \neq 0$

$$w(y_1, y_2) = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix}$$

$$= -2 - (2) = -4 \neq 0$$

~~Ex $y_1 = \cos mt$, $y_2 =$~~

can also write $y_1 = \cosh 2x$

$y_2 = \sinh 2x$

$$y_1 = \frac{e^{2x} + e^{-2x}}{2} = \cosh 2x$$

$$y_2 = \frac{e^{2x} - e^{-2x}}{2} = \sinh 2x$$

so general soln can either be written

as

$$y = c_1 e^{2x} + c_2 e^{-2x}$$

<u>or</u>

$$y = c_1 \cosh 2x + c_2 \sinh 2x$$

$y = c_1 \cos mt$
$+ c_2 4 \cos mt$
$= (c_1 + 4c_2)\cos mt$
$= A \cos mt$

<u>Ex</u> $y_1 = \cos mt$, $y_2 = 4\cos mt$

are soln's of $y'' = -m^2 y$ but

$$W(y_1, y_2) = \begin{vmatrix} \cos mt & 4\cos mt \\ -m\sin mt & -4m\sin mt \end{vmatrix}$$

$$= -4m \cos mt \sin mt - (-4m \cos mt \sin mt)$$

$$= 0$$

Abel's formula (show W is either always 0 or never 0).

$$W(y_1, y_2; x) = y_1 y_2' - y_2 y_1'$$

$$\left[\; y_{1,2}'' + p(x)\, y_{1,2}' + q(x)\, y_{1,2} = 0 \;\right]$$

$$\boxed{\frac{dW}{dx} = \cancel{y_1' y_2'} + y_1 y_2'' \cancel{- y_2' y_1'} - y_2 y_1''}$$

$$y_{1,2}'' = -p(x)\, y_{1,2}' - q(x)\, y_{1,2}$$

therefore
$$y_1'' = -p(x) y_1' - q(x) y_1$$
$$y_2'' = -p(x) y_2' - q(x) y_2$$

?

$$\frac{dW}{dx} = y_1\left[-p(x)\, y_2' - q(x)\, y_2\right]$$

$$- y_2\left[-p(x)\, y_1' - q(x)\, y_1\right]$$

$$= p(x)\left[-y_1 y_2' + y_2 y_1'\right] + q(x)\cancel{\left[-y_1 y_2 + y_2 y_1\right]}$$

$$= p(x)\,(-W)$$

$$\Rightarrow \frac{dw}{dx} = -p(x)\, W \qquad \text{(separable)}$$

$$\int \frac{dw}{w} = \int -p(x)\, dx$$

$$\log w = -\int p(x)\, dx + C$$

$$w = A e^{-\int p(x)\, dx}$$

↑ exp. is <u>never 0</u>

therefore, w is always 0 $(A = 0)$

or never 0 $(A \neq 0)$

<u>Linear 2nd order equations with constant coeff's</u>

have the form

$$ay'' + by' + cy = 0 \qquad (5)$$

where a, b, c are constants.

- since existence and uniquess theorem applies, we can guess at form of solution.

- notice that $\frac{d}{dx} e^{rx} = r e^{rx}$

that is : differentiation $\rightarrow$ multiplication

so using this guess in (5), we obtain an algebraic equation for r.

(ODE $\rightarrow$ algebraic equation)

<u>EX</u> solve

$$\frac{d^2}{dx^2} y = 4y \qquad (a=1,\ b=0,\ c=-4)$$

guess : $y = e^{rx}$

$$y' = r e^{rx}, \quad y'' = r^2 e^{rx}$$

$$r^2 e^{rx} = 4 e^{rx}$$

$$e^{rx}(r^2 - 4) = 0 \qquad e^{rx} \neq 0$$

D21

$\Rightarrow \quad r^2 - 4 = 0$ (characteristic equation C.E.)

~~r = ±2~~ $r_1 = 2, \quad r_2 = -2$

therefore

$y_1 = e^{2x}, \quad y_2 = e^{-2x}$ are

both solutions

so general soln can be written as

$$y = c_1 e^{2x} + c_2 e^{-2x}$$

or

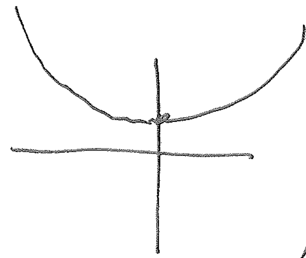
$$y = c_1 \cosh 2x + c_2 \sinh 2x$$

$$= c_1 \left[\frac{e^{2x} + e^{-2y}}{2} \right] + c_2 \left[\frac{e^{2x} - e^{-2x}}{2} \right]$$

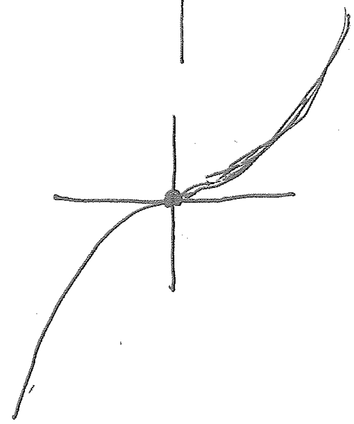
$$= e^{2x} \underbrace{\left[\frac{c_1}{2} + \frac{c_2}{2} \right]}_{A} + e^{-2x} \underbrace{\left[\frac{c_1}{2} - \frac{c_2}{2} \right]}_{B}$$

$$= A e^{2x} + B e^{-2x}$$ $\therefore$ equivalent.

$$\cosh x \equiv \frac{e^{x}+e^{-x}}{2}$$



$$\sinh x = \frac{e^{x}-e^{-x}}{2}$$



$$\cos x = \frac{e^{ix}+e^{-ix}}{2}$$

$$\sin x = \frac{e^{ix}-e^{-ix}}{2i}$$

$$\begin{array}{|l} \cosh^2 x - \sinh^2 x = 1 \\ \cos^2 x + \sin^2 x = 1 \end{array}$$

~~Ex (p.109)~~

$$W(y_1, y_2) = \begin{vmatrix} \cosh 2x & \sinh 2x \\ 2\sinh 2x & 2\cosh 2x \end{vmatrix}$$

$$= 2\cosh^2 2x - 2\sinh^2 2x$$

$$= 2\{\underbrace{\cosh^2 2x - \sinh^2 2x}_{1}] = 2$$

(homework)

Ex (p.109)

$$y'' - 5y' + 6y = 0$$

guess $y = e^{rx}$

$y' = re^{rx}$, $y'' = r^2 e^{rx}$

sub into ODE:

$r^2 e^{rx} - 5re^{rx} + 6e^{rx} = 0$

$\underbrace{e^{rx}}_{\text{never } 0} [r^2 - 5r + 6] = 0$

$\Rightarrow r^2 - 5r + 6 = 0$ (C.E.)

$(r-2)(r-3) = 0$

$r = 2, 3$

$y_1 = e^{2x}$, $y_2 = e^{3x}$

no sinh and cosh because e^{-2x} and e^{-3x} are not soln's of the ODE

go back to (5): guess e^{rx}

then C.E. would be

$$ar^2 + br + c = 0$$

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

the two previous examples were examples in which r_1 and r_2 were real and distinct. i.e., $\underline{b^2 - 4ac > 0}$ (case 1). Then we had

$$y_1 = e^{r_1 x}, \quad y_2 = e^{r_2 x}$$

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

what if $b^2 - 4ac < 0$? (case 2)

r_1 and r_2 would still be distinct but complex. so we can still write soln as

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

but r_1, r_2 are complex, so we can write in a more suggestive form

<u>EX</u> $y'' + 9y = 0$

$$r^2 + 9 = 0 \Rightarrow \cancel{r = \pm\sqrt{-3} = \pm 3i}$$

$$r^2 = -9 \Rightarrow r = \pm\sqrt{-9} = \pm 3i$$

$$y_1 = e^{3ix}, \quad y_2 = e^{-3ix}$$

so $y = c_1 e^{3ix} + c_2 e^{-3ix}$ is a general soln

<u>wronskian</u>

$$\begin{vmatrix} e^{i3x} & e^{-3ix} \\ 3ie^{3ix} & -3ie^{-3ix} \end{vmatrix}$$

$$= -3i - 3i = -6i \neq 0$$

can write in different form:

by superposition,

$$y = \frac{y_1 + y_2}{2}, \quad \text{and } y = \frac{y_1 - y_2}{2i}$$

are also soln's

$$\frac{y_1 + y_2}{2} = \frac{e^{i3x} + e^{-i3x}}{2} = \frac{\cos 3x + i\cancel{\sin 3x} + \cos 3x - i\cancel{\sin 3x}}{2}$$

$$= \cos 3x$$

$$\frac{y_1 - y_2}{2i} = \frac{e^{i3x} - e^{-i3x}}{2i} = \frac{\cancel{\cos 3x} + i\sin 3x - (\cancel{\cos 3x} - i\sin 3x)}{2i}$$

$$= \frac{2i\sin 3x}{2i} = \sin 3x$$

so $y_1 = \cos 3x$ and $y_2 = \sin 3x$

are l.i. soln's also

so $y = c_1 \cos 3x + c_2 \sin 3x$

is a general soln

notice: $\cos 3x = \mathrm{Re}\,(e^{i3x})$

$\sin 3x = \mathrm{Im}\,(e^{i3x})$

so the real and imaginary part of e^{i3x} form <u>two</u> linearly indep. soln's

(will use this fact later in the course)