Instability thresholds of mesa-type patterns in reaction-diffusion systems

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Introduction

Consider the system

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We consider the stability of these patterns by examining the eigenvalue problem.

Eigenvalues

We linearize about the steady state and obtain the following eigenvalue problem

$$\lambda\phi = \varepsilon^2\phi'' + f_u(u,w)\phi + f_w(u,w)\psi \quad (1)$$

$$0 = D\psi'' + g_u(u,w)\phi + g_w(u,w)\psi \quad (2)$$

with Neumann boundary conditions.

To analyze this, we first consider this problem with periodic boundary conditions on the interval $[-L, L]$:

$$\phi(L) = z\phi(-L), \;\; \phi'(L) = z\phi'(-L), \psi(L) = z\psi(-L), \;\; \psi'(L) = z\psi'(-L)$$

where $z = \exp\left(\frac{2\pi ik}{K}\right)$ for $k = 0, \ldots, K-1$.

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where $z = \exp\left(\frac{2\pi ik}{K}\right)$ for $k = 0, \ldots, K-1$.
Then this solution can be extended to the whole interval $[-L, (2K-1)L]$ with periodic boundary conditions.

Eigenvalues: Periodic Boundary Conditions

Consider one mesa on the interval $[-L, L]$. Denote the interface locations by ℓ. The eigenfunctions are estimated as

$$\phi \sim c_{\pm} u_x, \quad \psi \sim \psi(\pm\ell), \quad \text{when } x \sim \pm\ell$$

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Multiplying (1) by u_x and integrating over $[-L, 0]$, as well as using the eigenfunction estimates, we obtain

$$\lambda c_- \int_{-L}^{0} u_x^2 dx \sim \varepsilon^2 \left(\psi_x u_x - \phi u_{xx}\right)\Big|_{-L}^{0} + \left(\psi(-\ell) - c_- w_x(-\ell)\right) \int_{u_-}^{u_+} f_w du$$

Similarly, on the interval $[0, L]$

$$\lambda c_+ \int_{0}^{L} u_x^2 dx \sim \varepsilon^2 \left(\psi_x u_x - \phi u_{xx}\right)\Big|_{0}^{L} - \left(\psi(\ell) - c_+ w_x(\ell)\right) \int_{u_-}^{u_+} f_w du$$

Eigenvalues: Periodic Boundary Conditions

Simplifying

$$\lambda\kappa_0 \begin{pmatrix} c_+ \\ c_- \end{pmatrix} = \begin{pmatrix} \kappa_1 \left(-\phi u_{xx}\right)_0^L - \psi(\ell) + c_+ w_x(l) \\ \kappa_1 \left(-\phi u_{xx}\right)_{-L}^0 + \psi(-\ell) - c_- w_x(-\ell) \end{pmatrix}$$

where

$$\kappa_0 = \frac{\int_{-L}^0 u_x^2 dx}{\int_{u_-}^{u_+} f_w du}, \quad \kappa_1 = \frac{\varepsilon^2}{\int_{u_-}^{u_+} f_w du}$$

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Next, we determine the boundary terms and ψ terms, then solve for λ in terms of D and $z = e^{i\theta}, \theta = \frac{2\pi k}{K}, k = 1, \ldots, K-1$

Eigenvalues: Periodic Boundary Conditions Result

$$\lambda_\theta^\pm \sim (a + |b|) \frac{\int_{u_-}^{u_+} f_w \, du}{\int_0^{-L} u_x^2 \, dx}$$

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where

$$\begin{aligned} a &= \alpha_+ + \alpha_- + \frac{(g_+ - g_-)}{D} \frac{L}{1 - \cos\theta} - \frac{g_+ \ell}{D} \\ |b|^2 &= \alpha_+^2 + \alpha_-^2 + 2\alpha_+\alpha_- \cos\theta + \frac{2(g_+ - g_-)}{D} \left[\frac{L(\alpha_+ + \alpha_-)}{1 - \cos\theta} - \ell\alpha_+ - (L - \ell)\alpha_- \right] \\ &\quad + \frac{(g_+ - g_-)^2}{D^2 (1 - \cos\theta)^2} \left[L^2 - 2(1 - \cos\theta)\ell(L - \ell) \right] \end{aligned}$$

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with

$$\alpha_+ = \frac{2C_+^2 \mu_+^3}{\int_{u_-}^{u_+} f_w du} \frac{1}{\varepsilon} \exp\left(-\frac{2\mu_+}{\varepsilon} \ell \right); \alpha_- = \frac{2C_-^2 \mu_-^3}{\int_{u_-}^{u_+} f_w du} \frac{1}{\varepsilon} \exp\left(-\frac{2\mu_-}{\varepsilon} (L - \ell) \right)$$

$$\theta = \frac{2\pi k}{K}, \ k = 1, .., K - 1$$

Eigenvalues: Neumann Boundary Conditions

Consider the steady state consisting of K mesas on the interval of size $2KL$, with Neumann boundary conditions. The linearized problem admits $2K$ eigenvalues. These are those given by the periodic boundary conditions and the following

$$\lambda_{even} = -\frac{g_- - g_+}{\sigma_+ \ell + \sigma_-(L-\ell)} \frac{\int_{u_-}^{u_+} f_w du}{\int_0^{-L} u_x^2 dx}$$

and

$$\lambda_{odd} = \left(2\alpha_- - \frac{g_-^2 L}{D(g_- - g_+)}\right) \frac{\int_{u_-}^{u_+} f_w du}{\int_0^{-L} u_x^2 dx}$$

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To determine the critical value of D, where the mesa patterns become unstable, we set $\lambda_\theta = 0$ and solve, giving

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$$D_\theta = \frac{L}{2(g_- - g_+)(g_-^{-2}\alpha_- + g_+^{-2}\alpha_+)} \left(\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{2\alpha_+\alpha_-(1-\cos\theta)g_+^2 g_-^2}{4(g_-^2\alpha_+ + g_+^2\alpha_-)^2}} \right)^{-1}$$

Stability Thresholds

Since this function D_θ is an increasing function of θ, the first eigenvalue to become unstable corresponds to $k = 1$ where $k = 1, \ldots K - 1$.

$$D_K = \frac{L}{2(g_- - g_+)(g_-^{-2}\alpha_- + g_+^{-2}\alpha_+)} \left(\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{2\alpha_+\alpha_-(1 - cos\pi/K)g_+^2 g_-^2}{4(g_-^2\alpha_+ + g_+^2\alpha_-)^2}} \right)^{-1}$$

Simple example

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f(u,w) &= 2(u-u^2)+w \\
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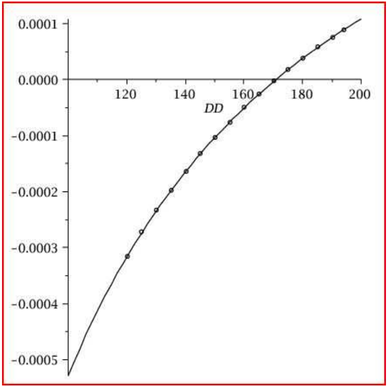
$\varepsilon = 0.17,\ L = 1,\ \ell = \frac{1+\beta_0}{2},\ \beta_0 = 0$

Initial conditions:

$$u(x,0) = \tanh\left(\frac{(x-x_0)+\ell}{\varepsilon}\right) - \tanh\left(\frac{(x-x_0)-\ell}{\varepsilon}\right) - 1, \quad w(x,0) = 0$$

Numerics

Circles give λ computed from solving the eigenvalue problem numerically and the solid line gives the asymptotic approximation to λ.



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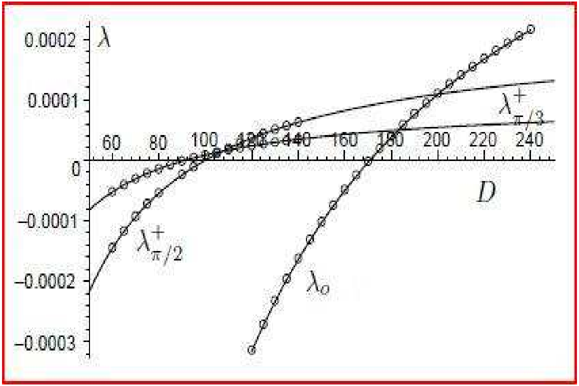

Thank you!