Mesa-type patterns in Reaction Diffusion Systems

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The Problem

Consider the problem

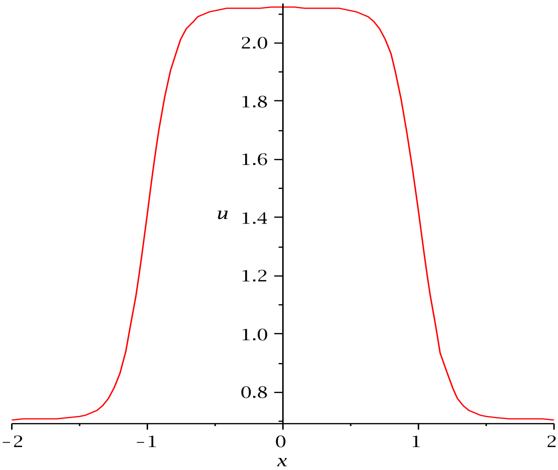
$$\begin{aligned} u_t &= \varepsilon^2 u_{xx} + f(u,w) \\ w_t &= D w_{xx} + g(u,w) \end{aligned}$$

with Neumann boundary conditions and $x \in [-L, L]$ where

$$\begin{aligned} f(u,w) &= -u + u^2(w-u) \\ g(u,w) &= 1 - \beta_0 u. \end{aligned}$$

Here D is exponentially large, and ε is small.
We want to consider the stability of patterns in the profile of u.

One Mesa



Equations of motion of the interfaces

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We let l_- and l_+ be the interfaces, $-L < l_- < l_+ < L$.
Let $u = u_-(x - l_-)$ on $(-L, x_0)$ and $u = u_+(x - l_+)$ on (x_0, L).

Expand $u = u_0 + \frac{1}{D} u_1$ and $w = w_0 + \frac{1}{D} w_1$. Note w_0 is a constant.

On $(-L, x_0)$, we define $u_0(x) = u_-(x - l_-) = U_-\left(\frac{x-l_-}{\varepsilon}\right)$.
Similarly, on (x_0, L), $u_0(x) = u_+(x - l_+) = U_+\left(\frac{x-l_+}{\varepsilon}\right)$

Equations of motion of the interfaces

Then

$$-\varepsilon l_-' \int_{-\infty}^{\infty} (U_-')^2 ds = \frac{\varepsilon^2}{D} \left[u_1' u_0' - u_1 u_0''\right]\Big|_{-L}^{x_0} + \frac{1}{D} w_1(l_-) \int_0^{\sqrt{2}} f_w \, dU_-.$$

$$-\varepsilon l_+' \int_{-\infty}^{\infty} (U_+')^2 \, ds = \frac{\varepsilon^2}{D} \left[u_1' u_0' - u_1 u_0''\right]\Big|_{x_0}^{L} - \frac{1}{D} w_1(l_+) \int_0^{\sqrt{2}} f_w \, dU_+.$$

Since $x_0 = \frac{l_+ + l_-}{2}$,

$$\frac{dx_0}{dt} = \frac{\varepsilon}{2} \frac{1}{\int_{-\infty}^{\infty} (U'(s))^2 \, ds} \Bigg\{ -\frac{\varepsilon^2}{D} \Big[u_1' u_-' - u_1 u_-'' \Big|_{-L}^{x_0}$$

$$+ \; (u_1' u_+' - u_1 u_+'')\Big|_{x_0}^{L} \Big] + \frac{1}{D} (w_1(l_+) - w_1(l_-)) \int_0^{\sqrt{2}} f_w \, dU \Bigg\}.$$

The boundary terms are determined to be

$$u_1' u_-' - u_1 u_-'' \Big|_{-L}^{x_0} + u_1' u_+' - u_1 u_+'' \Big|_{x_0}^{L}$$

$$= 2\frac{D}{\varepsilon^2}\mu_0^2 C_0^2 \left(e^{\frac{\mu_0}{\varepsilon}(-2x_0+d-2L)} - e^{\frac{\mu_0}{\varepsilon}(2x_0+d-2L)} \right)$$

where d is the width of the mesa and is given by $d = \frac{\sqrt{2}}{\beta_0} L$, where $l = d/2$, $f_w = f_w(u, w_0)$, and μ_0, C_0 are constants.

As well, it is determined that

$$w_1(l_+) - w_1(l_-) = -2x_0 l g(0, w_0).$$

Then, the equation of motion for x_0 is

$$\frac{dx_0}{dt} = \frac{\varepsilon}{\int_{-\infty}^{\infty}(U'(s))^2\ ds}\left\{\mu_0^2 C_0^2 e^{\frac{\mu_0}{\varepsilon}(d-2L)}\left[e^{\frac{\mu_0}{\varepsilon}2x_0} - e^{-\frac{\mu_0}{\varepsilon}2x_0}\right]\right.$$

$$\left. + \frac{1}{D}[-x_0 lg(0, w_0)]\int_0^{\sqrt{2}} f_w\ dU\right\}$$

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Critical Value of D

A change in stability of the differential equation occurs when

$$D_c = \frac{lg(0, w_0)\int_0^{\sqrt{2}} f_w \, dU}{4\frac{\mu_0^3}{\varepsilon} C_{0+}^2 e^{\frac{\mu_0}{\varepsilon}(d-2L)}}.$$

Substituting in all of the constants, we obtain the equation for D_c as a function of ε and L.

$$D_c = \frac{1}{12\beta_0} L\varepsilon \exp(\frac{1}{\varepsilon}(2 - \frac{\sqrt{2}}{\beta_0})L).$$

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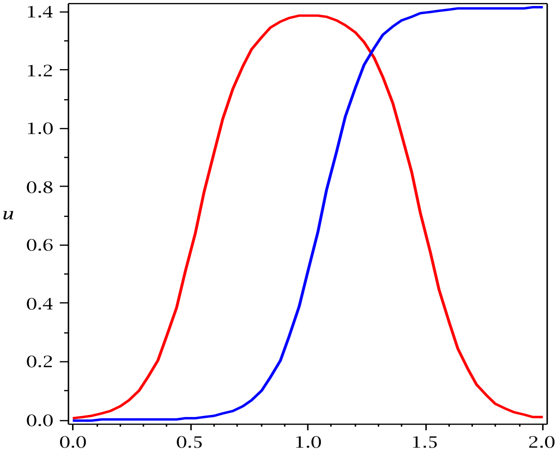
Once D has been increased beyond this value, the pattern becomes unstable.

Numerical Simulation of Full System

$$\varepsilon = 0.1,\ \beta_0 = 1.5,\ L = 1,\ D = 2000$$



Comparison of x_0' and Solution from Full System

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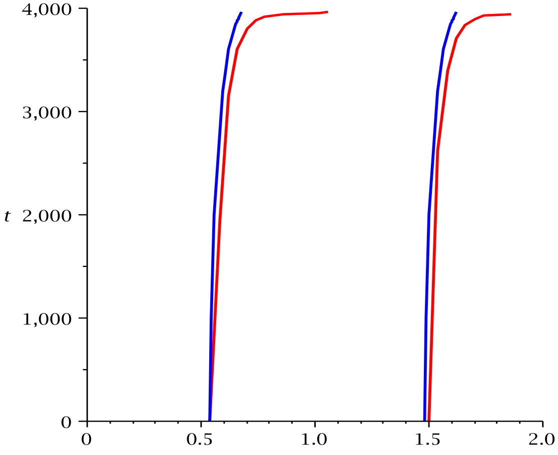

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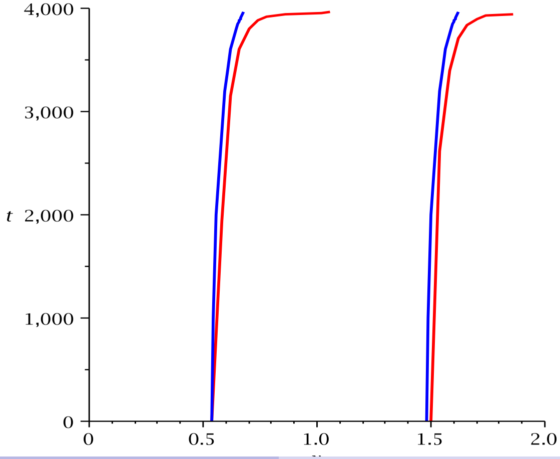
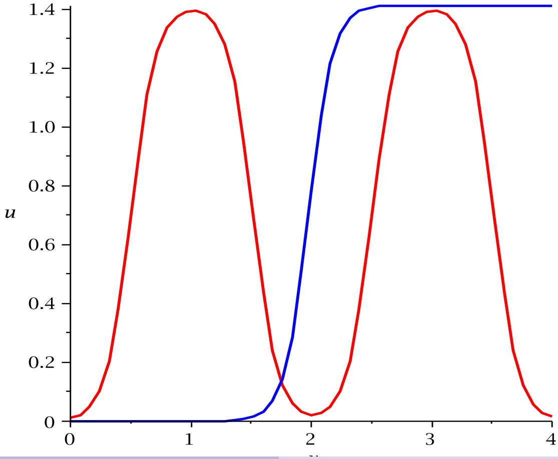


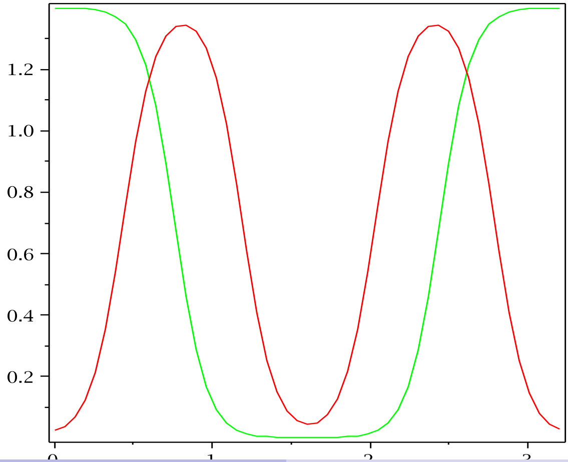
Profiles of Two Mesas

$$D = 2000,\ L = 1,\ \varepsilon = 0.1,\ \beta_0 = 1.4$$



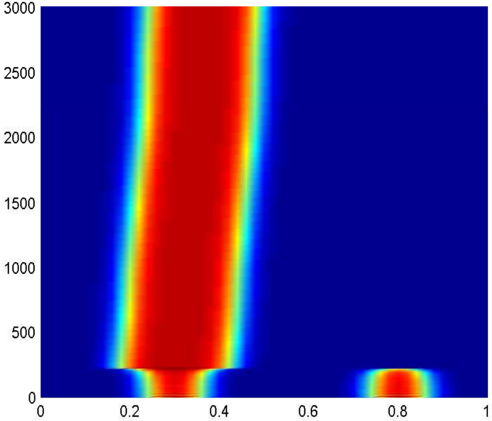
Multiple Mesas

$$D = 30,\ L = 0.8,\ \varepsilon = 0.1,\ \beta_0 = 1.5$$



Profiles of Two Mesas

$$D = 28.28, \; L = 1, \; \varepsilon = 0.0177, \; \beta_0 = 2.828$$



Multiple Mesas

Multiple mesas can exhibit different types of behaviour, depending on which interface becomes unstable first.

Considering the solution of multiple mesas, similar analysis can be completed but becomes much more complicated.

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We look at the eigenvalues of the system.

Eigenvalues

Let (u_e, w_e) be the equilibrium solution. Let

$$\begin{aligned} u(x,t) &= u_e(x) + e^{\lambda t}\phi(x) \\ w(x,t) &= w_e(x) + e^{\lambda t}\psi(x) \end{aligned}$$

and then substitute into the system. This gives the following:

$$\lambda\phi = \varepsilon^2\phi_{xx} + \phi f_u(u_e, w_e) + \psi f_w(u_e, w_e).$$

Similarly, we obtain

$$\lambda\psi = D\psi_{xx} + \phi g_u(u_e, w_e) + \psi g_w(u_e, w_e).$$

Eigenvalues

The eigenvalues are given by the following expression:

$$\frac{1}{3\varepsilon}\lambda\begin{bmatrix} C_l \\ -C_r \end{bmatrix} = B\begin{bmatrix} C_l \\ -C_r \end{bmatrix} + \frac{(\sqrt{2})^3}{3}M\begin{bmatrix} C_l \\ -C_r \end{bmatrix} + \frac{(\sqrt{2})^3}{3}\frac{1}{D}\frac{1}{2}(1-\frac{1}{\sqrt{2}\beta_0})L\begin{bmatrix} C_l \\ -C_r \end{bmatrix}$$

where B is the matrix determined by the boundary conditions and M^{-1} is the matrix determined from the ψ terms.
This gives the eigenvalues as

$$\lambda_\pm = 3\varepsilon\left[\eta_\pm + \frac{(\sqrt(2))^3}{3}\frac{1}{\sigma_\pm} + \frac{(\sqrt{2})^3}{3}\frac{1}{D}\frac{1}{2}\left(1-\frac{1}{\sqrt{2}\beta_0}\right)L\right]$$

where $\sigma_\pm$ are the eigenvalues of the matrix M^{-1} and where $\eta_\pm$ are the eigenvalues of the matrix B.

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This simply changes the $\eta_\pm$ and $\sigma_\pm$ terms of the previous expression.

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We can compare these asymptotic eigenvalues by determining the eigenvalues numerically.

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Rewriting our system, with the equations for the eigenfunction as a BVP, we can compute the eigenvalues numerically in Maple.

Work in progress

For multiple mesa patterns, the ways the patterns become unstable are believed to be caused by which eigenvalue changes from negative to positive first.

Once we have verified that these eigenvalues agree with the numerically determined eigenvalues, a more careful analysis will hopefully lead to a general theory of which eigenvalues lead to the different ways that the patterns can become unstable.