Stability of a Reaction-Diffusion Model with Mesa-type Patterns

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Asymptotic analysis of localized patterns in PDEs (CAIMS)

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Introduction

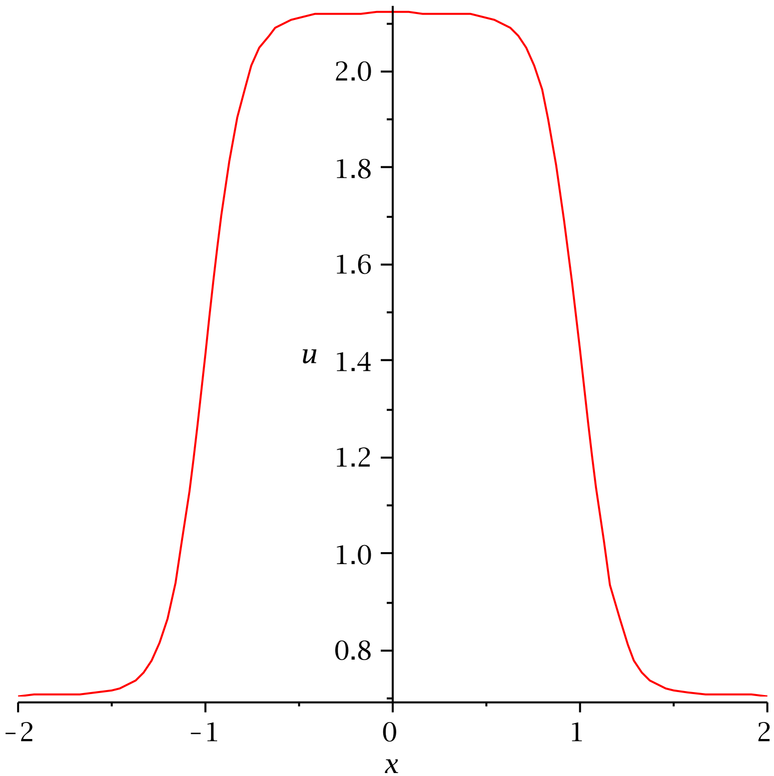
Consider the reaction-diffusion system

$$\begin{aligned} u_t &= \varepsilon^2 u_{xx} + f(u,w) \\ w_t &= D w_{xx} + g(u,w) \end{aligned}$$

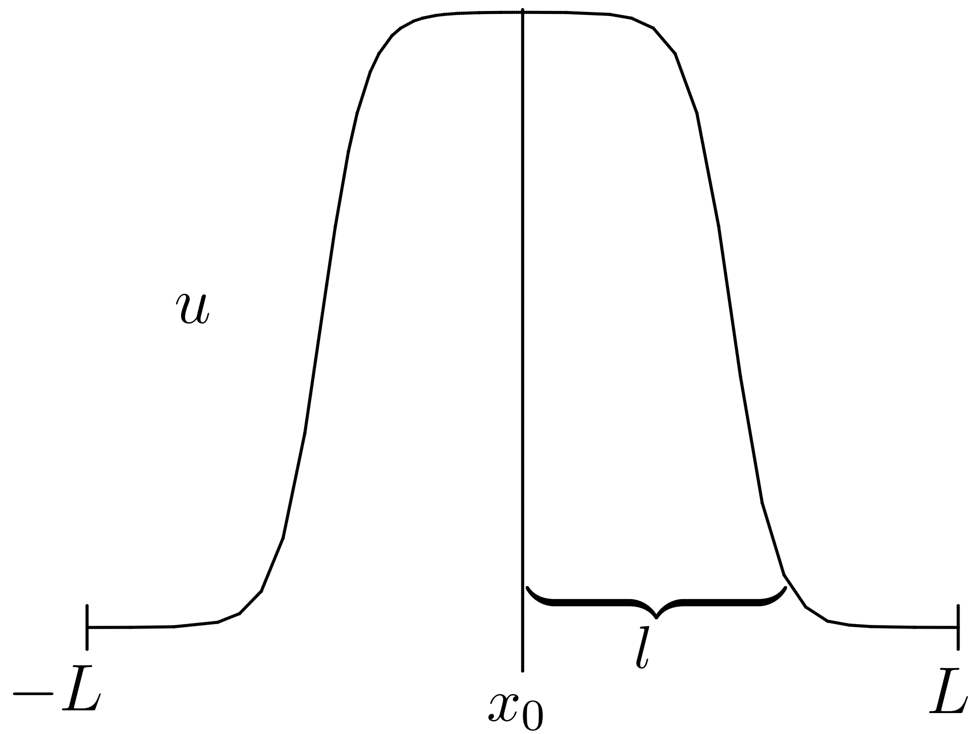
where $\varepsilon << 1$ and $D >> \varepsilon$ with Neumann boundary conditions and $x \in [-L, L]$, where

$$\begin{aligned} f(u,w) &= -u + u^2(w-u) \\ g(u,w) &= 1 - \beta_0 u. \end{aligned}$$

In particular, we consider solutions of u with sharp interfaces, giving *mesas* such as those given to the right, and then examine the motion of these interfaces.



Part A: Determining the Equation of Motion for One Mesa



Let $u = u_-(x - l_-)$ on $(-L, x_0)$ and $u = u_+(x - l_+)$ on (x_0, L) where l_- and l_+ are interfaces, $-L < l_- < l_+ < L$. Assume that the interfaces move slowly in time, $l_- = l_-(\varepsilon^2 t)$ and $l_+ = l_+(\varepsilon^2 t)$. The width of the mesa is a constant (due to conservation of mass). We can define

$$x_0 = \frac{l_+ + l_-}{2}.$$

Note that $u'(x_0) = 0$.

Expand $u = u_0 + \frac{1}{D}u_1$ and $w = w_0 + \frac{1}{D}w_1$. Note w_0 is a constant.

On $(-L, x_0)$, we define $u_0(x) = u_-(x - l_-) = U_-\left(\frac{x-l_-}{\varepsilon}\right)$.

Substituting the expansions in to our equations, we obtain the following

$$\begin{aligned}
-\varepsilon^2 l'_- u'_- &= \varepsilon^2 u_{0xx} + \varepsilon^2 \frac{1}{D} u_{1xx} + f(u_0, w_0) + \frac{1}{D} f_u u_1 + \frac{1}{D} f_w w_1 \\
0 &= D w_{0xx} + w_{1xx} + g(u_0, w_0) + \frac{1}{D} g_u u_1 + \frac{1}{D} g_w w_1.
\end{aligned}$$

Multiplying by u_{0x}, then integrating by parts gives

$$-\varepsilon l'_- \int_{-\infty}^{\infty} (U'_-)^2 ds = \frac{\varepsilon^2}{D} \left[u'_1 u'_0 - u_1 u''_0\right]\Big|_{-L}^{x_0} + \frac{1}{D} w_1(l_-) \int_0^{\sqrt{2}} f_w dU_-.$$

Similarly, on (x_0, L), $u_0(x) = u_+(x - l_+) = U_+\left(\frac{x-l_+}{\varepsilon}\right)$ and we obtain

$$-\varepsilon l'_+ \int_{-\infty}^{\infty} (U'_+)^2 \, ds = \frac{\varepsilon^2}{D} \left[u'_1 u'_0 - u_1 u''_0\right]\Big|_{x_0}^{L} - \frac{1}{D} w_1(l_+) \int_0^{\sqrt{2}} f_w dU_+.$$

Since $x_0 = \frac{l_+ + l_-}{2}$, we add together to equations for the two interfaces to obtain:

$$\frac{dx_0}{dt} = \frac{\varepsilon}{2} \frac{1}{\int_{-\infty}^{\infty} (U'(s))^2 \, ds} \left\{ - \frac{\varepsilon^2}{D} \left[u'_1 u'_- - u_1 u''_- \Big|_{-L}^{x_0} + (u'_1 u'_+ - u_1 u''_+)\Big|_{x_0}^{L} \right] \right.$$

$$\left. + \frac{1}{D} (w_1(l_+) - w_1(l_-)) \int_0^{\sqrt{2}} f_w \, dU \right\}.$$

The boundary terms are determined to be

$$u_1' u_-' - u_1 u_-'' \Big|_{-L}^{x_0} + u_1' u_+' - u_1 u_+'' \Big|_{x_0}^{L}$$

$$= 2\frac{D}{\varepsilon^2}\mu_0^2 C_0^2 \left(e^{\frac{\mu_0}{\varepsilon}(-2x_0+d-2L)} - e^{\frac{\mu_0}{\varepsilon}(2x_0+d-2L)} \right)$$

where d is the width of the mesa and is given by $d = \frac{\sqrt{2}}{\beta_0}L$,where $l = d/2$, $f_w = f_w(u, w_0)$, and μ_0, C_0 are constants.

As well, it is determined that

$$w_1(l_+) - w_1(l_-) = -2x_0 l g(0, w_0).$$

Then, the equation of motion for x_0 is

$$\frac{dx_0}{dt} = \frac{\varepsilon}{\int_{-\infty}^{\infty} (U'(s))^2 \, ds} \left\{ \mu_0^2 C_0^2 e^{\frac{\mu_0}{\varepsilon}(d-2L)} \left[e^{\frac{\mu_0}{\varepsilon} 2x_0} - e^{-\frac{\mu_0}{\varepsilon} 2x_0} \right] \right.$$

$$\left. + \frac{1}{D} [-x_0 l g(0, w_0)] \int_0^{\sqrt{2}} f_w \, dU \right\}.$$

Part A: Critical Value of D

A change in stability of the equilibrium of the differential equation, $x_0'(t)$, occurs when the diffusion coefficient, D, is

$$D = D_c = \frac{lg(0, w_0) \int_0^{\sqrt{2}} f_w \, dU}{4 \frac{\mu_0^3}{\varepsilon} C_{0+}^2 e^{\frac{\mu_0}{\varepsilon}(d-2L)}}$$

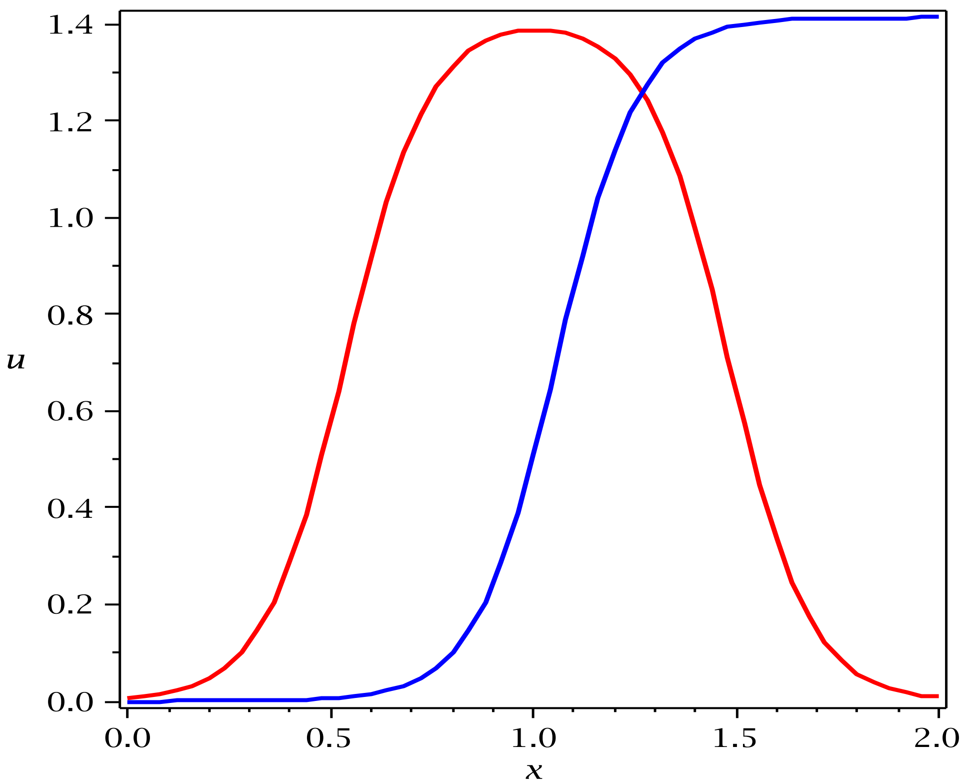
for general functions f and g. The interfaces will move when $D > D_c$.

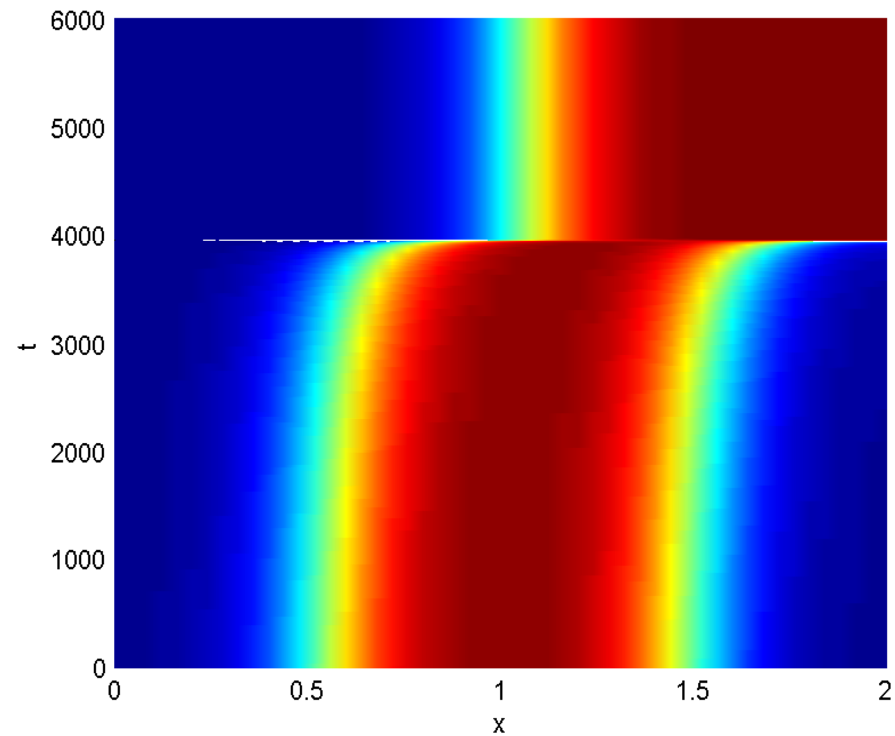
Substituting in the constants, we obtain the equation for D_c as a function of ε and L:

$$D_c = \frac{1}{12\beta_0} L\varepsilon \exp\left(\frac{1}{\varepsilon}\left(2 - \frac{\sqrt{2}}{\beta_0}\right)L\right).$$

Part A: Numerical Simulation of Full System

For the choice of parameters $\varepsilon = 0.1$, $\beta_0 = 1.5$, $L = 1$, and $D = 2000$, the numerical simulation of the system generates the following solutions.



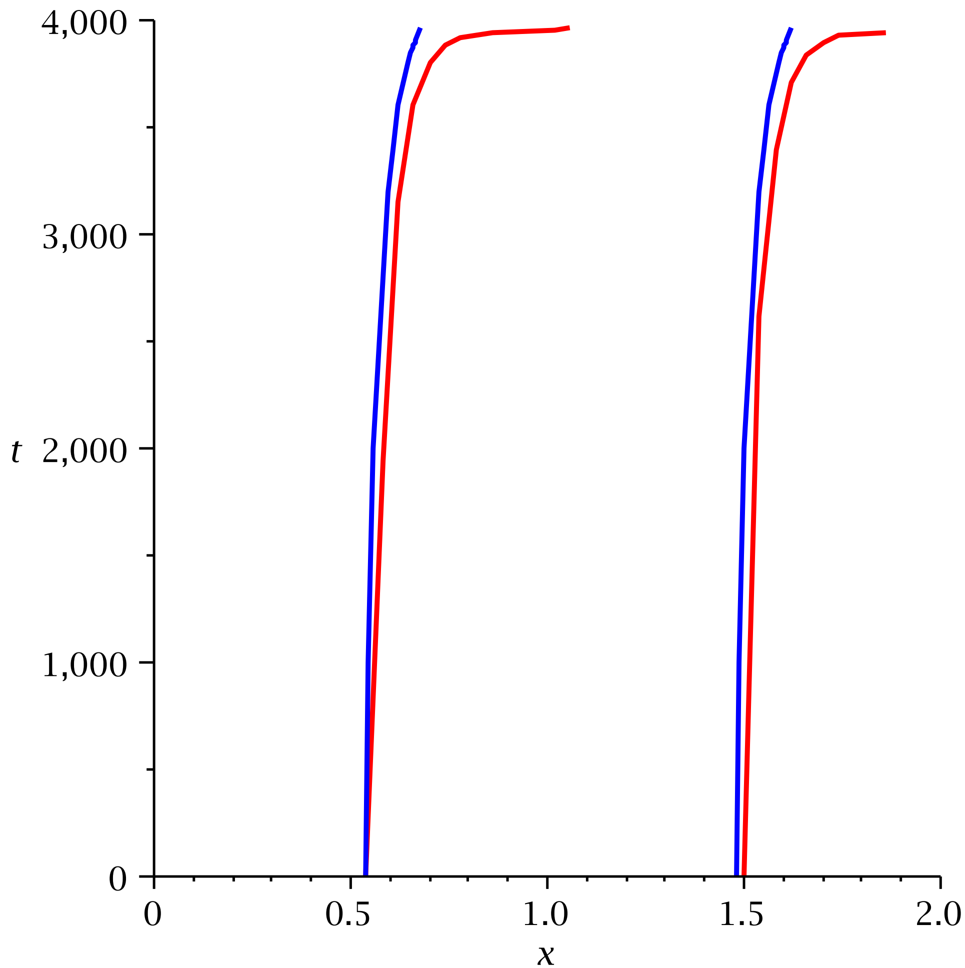


The red line denotes the initial profile of the u solution, the blue line denotes the final profile.

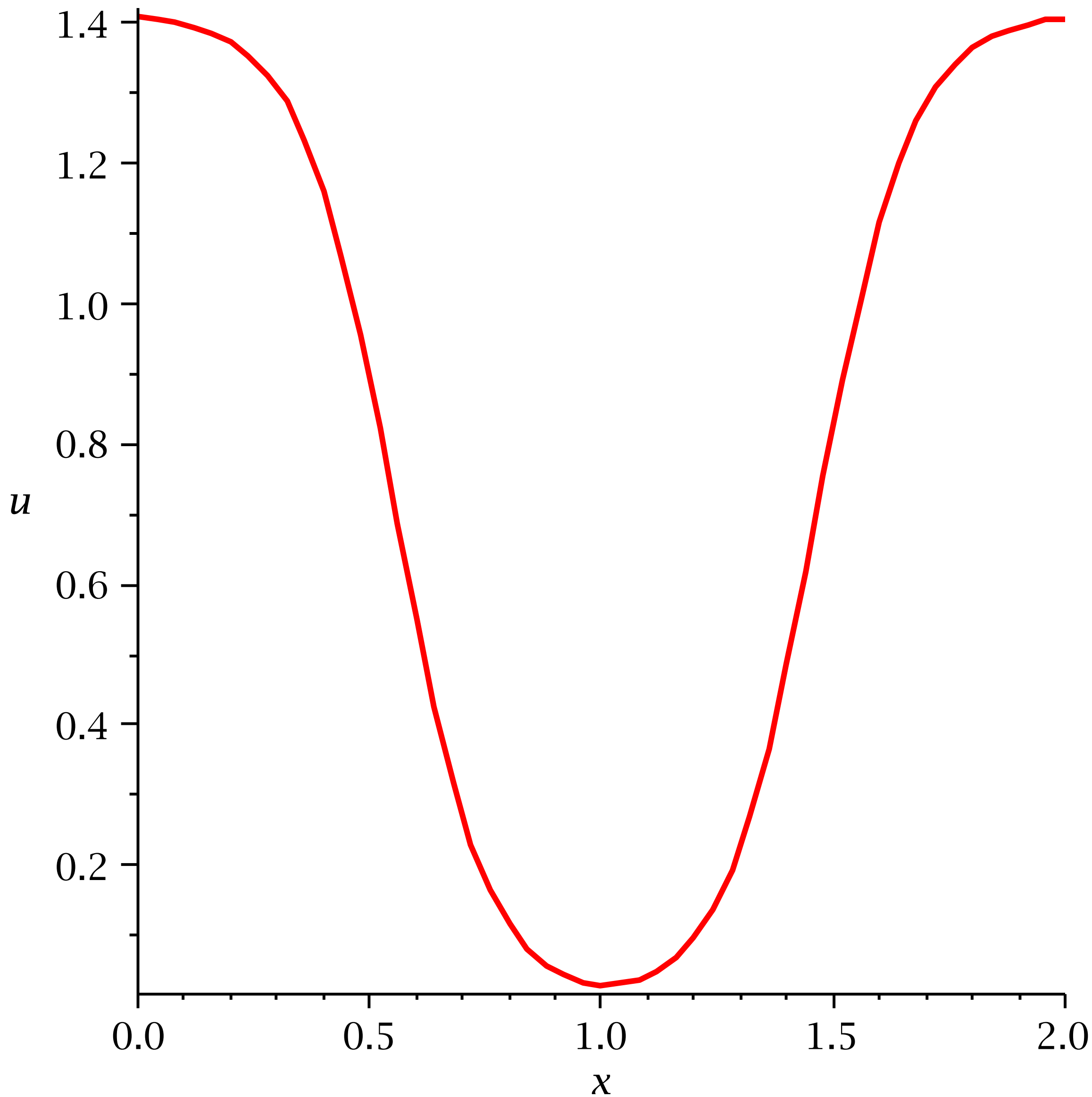
Blue is $u = 0$ and red is $u = \sqrt{2}$.

Comparison of x_0' and Solution of Full System

For the parameters $D = 2000$, $L = 1$, $\varepsilon = 0.1$, and $\beta_0 = 1.5$, we plot the motion of the point where $u = \frac{\sqrt{2}}{2}$ in time as obtained from the full numerical solution of the system (red) and from the determined ODE, x_0' (blue).



Part B: Determining the Equation of Motion Between Mesas



We proceed now just as we did for Part A. Once more, the interfaces are denoted as $l_- = l_-(\varepsilon^2 t)$ and $l_+ = l_+(\varepsilon^2 t)$.
The equation of motion for x_0 where

$$x_0 = \frac{(l_+ + l_-)}{2}$$

is given by

$$\frac{dx_0}{dt} = \frac{\varepsilon}{\int_{-\infty}^{\infty} (U'(s))^2 \, ds} \left\{ \mu_1^2 C_1^2 e^{\frac{\mu_1}{\varepsilon}(d-2L)} \left(e^{2\frac{\mu_1}{\varepsilon} x_0} - e^{-2\frac{\mu_1}{\varepsilon} x_0} \right) \right.$$

$$\left. + \frac{1}{D} [x_0 l g_+] \int_{-1}^{1} f_w \, dU \right\}$$

where d is the width between interfaces ($d = \frac{(\sqrt{2}\beta_0 - 1)2L}{\sqrt{2}\beta_0}$), $g_+ = g(\sqrt{2}, 0)$, $l = d/2$, $f_w = f_w(u, w_0)$, and μ_1, C_1 are constants.

Part B: Critical D Value

As we did for Part A, we obtain the following critical threshold for Part B,

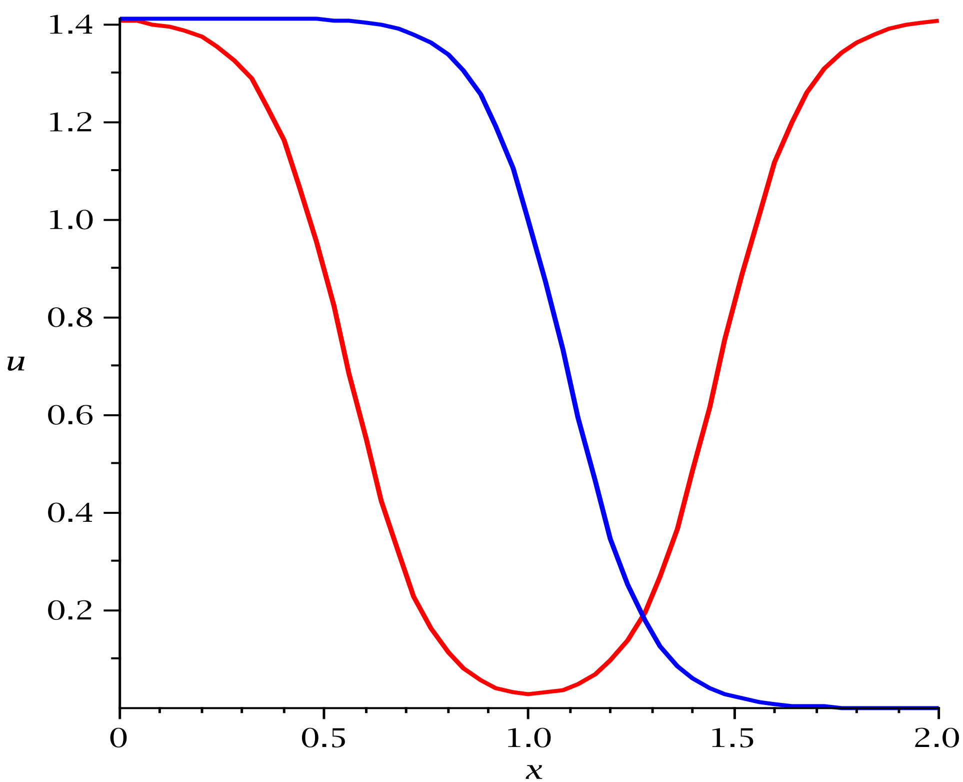
$$D_c = -\frac{l g_+ \int_0^{\sqrt{2}} f_w \, dU}{4\frac{\mu_1^3}{\varepsilon} C_1^2 e^{\frac{\mu_1}{\varepsilon}(d-2L)}}.$$

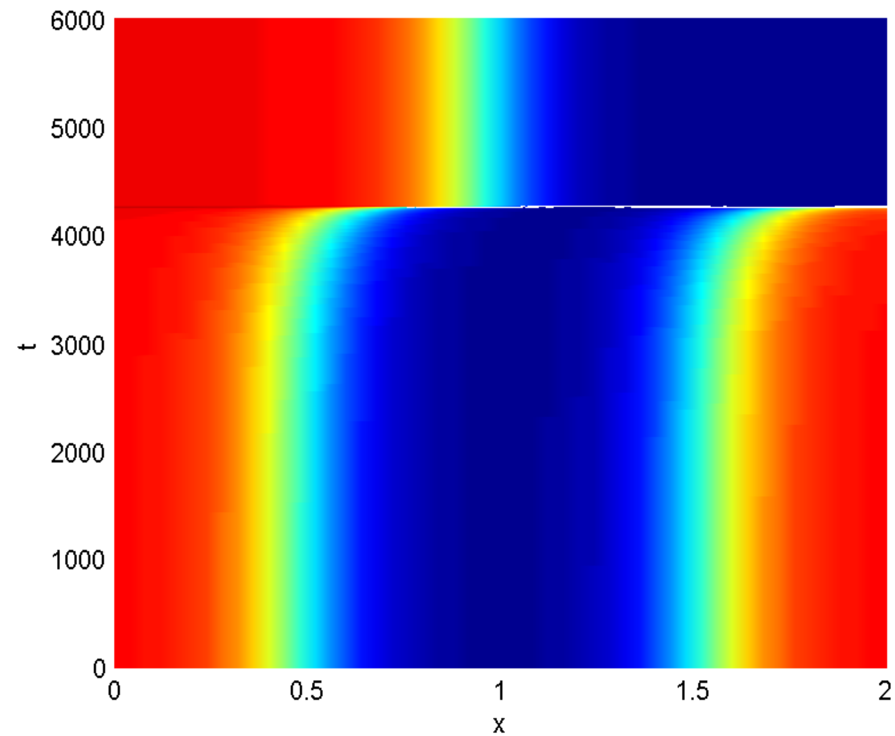
Substituting in constants that related to our particular model, we have

$$D_c = \varepsilon L \frac{(\sqrt{2}\beta_0 - 1)^2}{12\beta_0} \exp\left(\frac{L}{\varepsilon}\frac{\sqrt{2}}{\beta_0}\right).$$

Part B: Numerical Simulation of Full System

For the choice of parameters $\varepsilon = 0.1$, $\beta_0 = 1.5$, $L = 1$, and $D = 2000$, the numerical simulation of the system generates the following solutions.



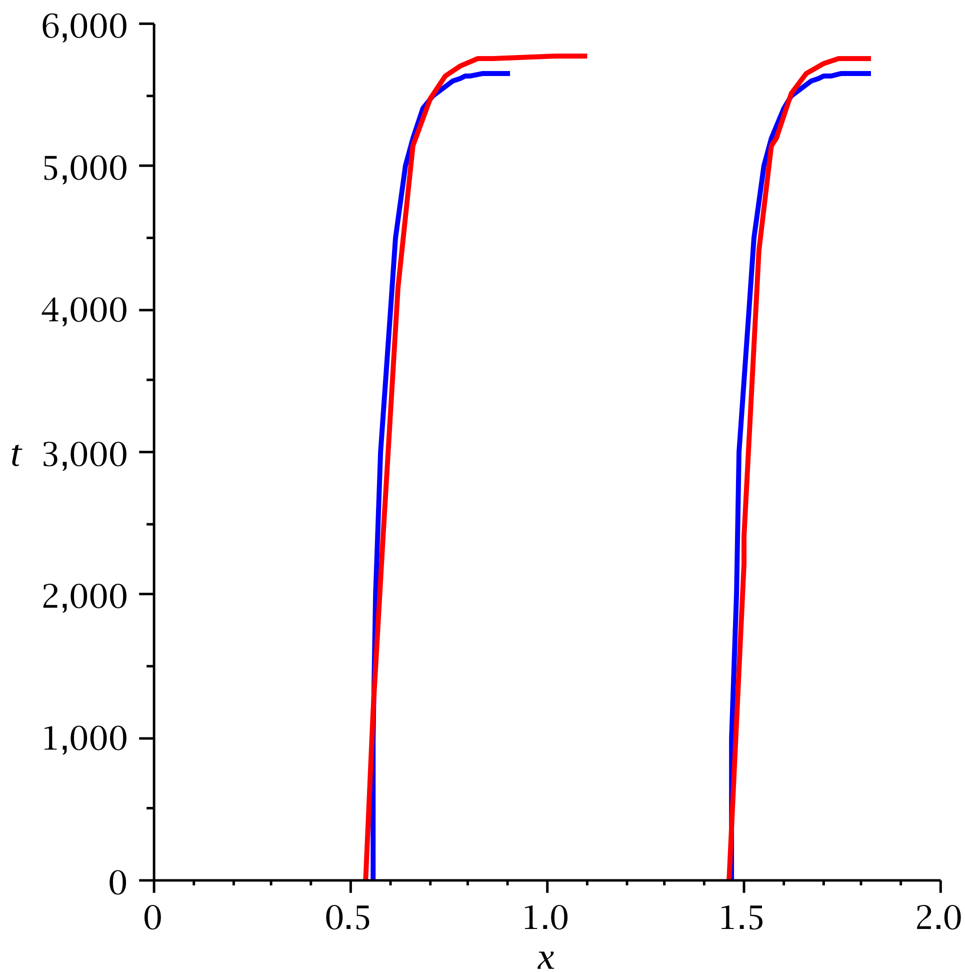


The red line denotes the initial profile of the u solution, blue line denotes the final profile.

Blue is $u = 0$ and red is $u = \sqrt{2}$.

Comparison of x_0' and Solution of Full System

For the parameters $D = 2000, L = 1, \varepsilon = 0.1$, and $\beta_0 = 1.3$, we plot the motion of the point where $u = \frac{\sqrt{2}}{2}$ in time as obtained from the full numerical solution of the system (red) and from the determined ODE, x_0', (blue).



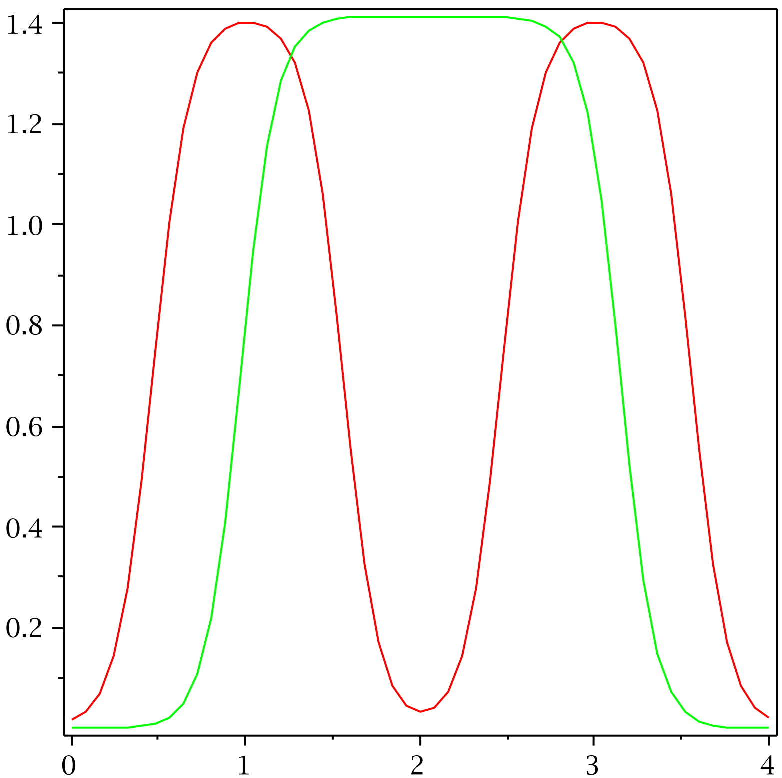
Dynamics of Two Mesas

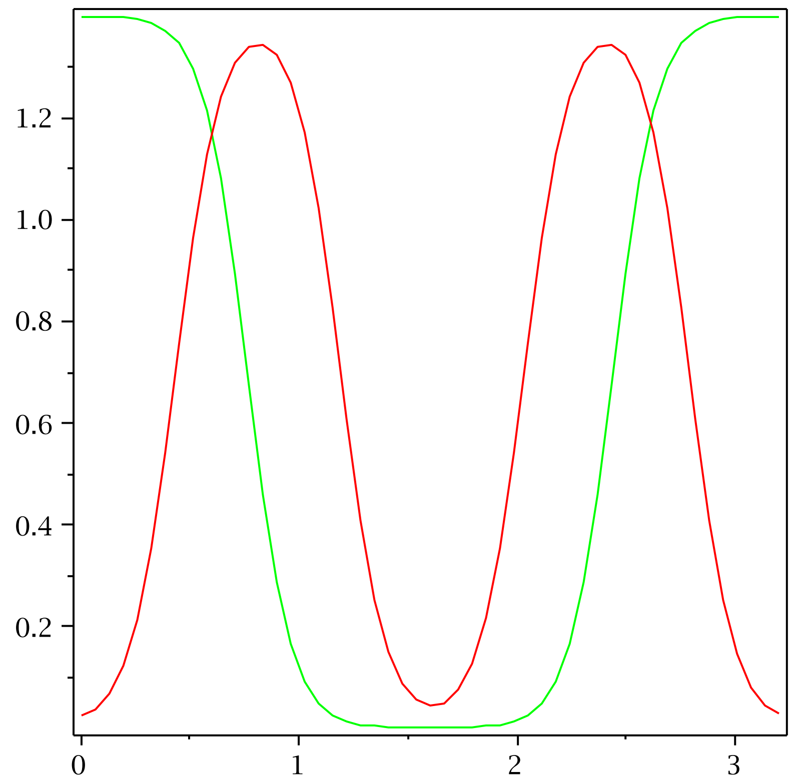
Now, we consider two mesas. A similar analysis can be completed, but is much more complicated because mass can be transferred between mesas. Also, different types of behaviour are exhibited: two mesas move towards each other, two mesas move away from each other, or all interfaces can move.

For some parameter choices, the threshold determined for Part B also gives the critical value for D for two mesas.

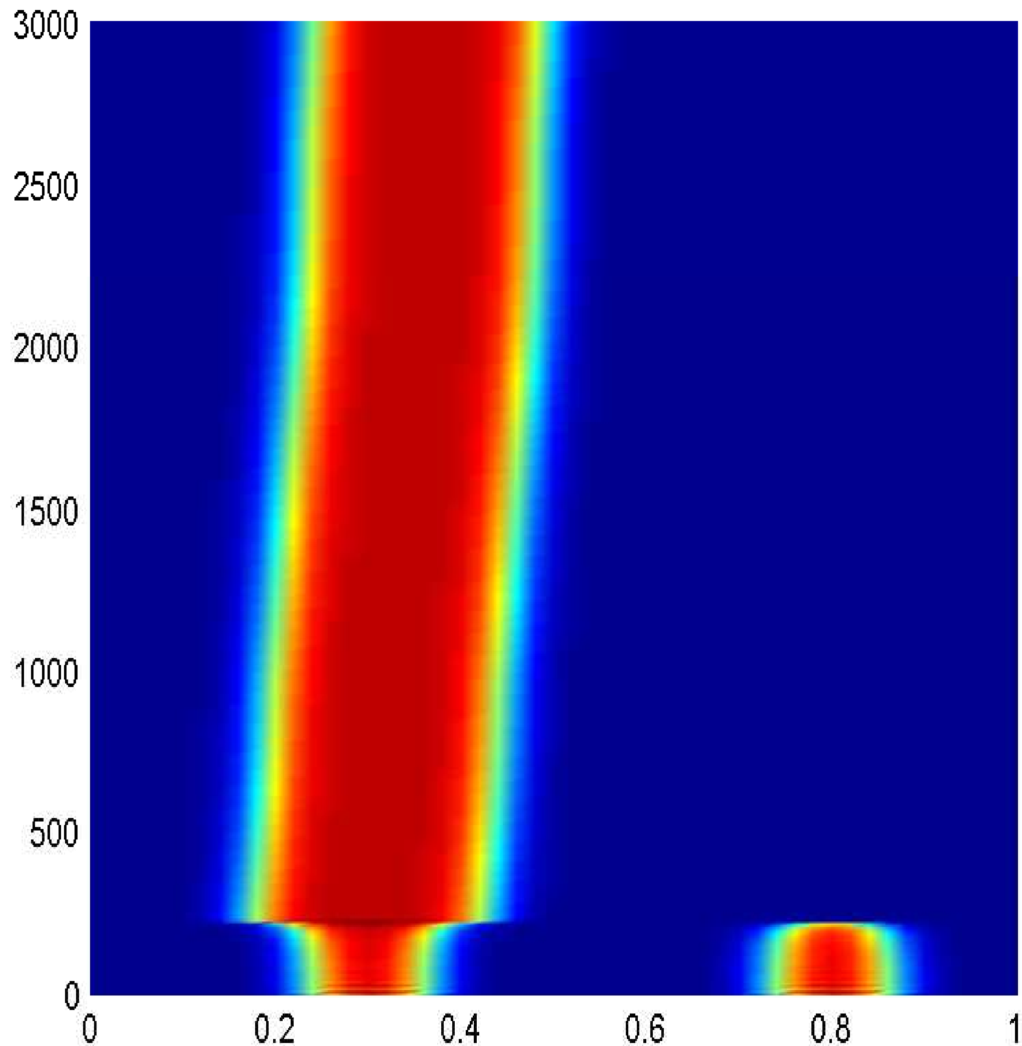
For some other parameter choices, the analysis in Part B can not describe the behaviour (such as two mesas moving towards each other).

Some examples of final profiles that are not possible to obtain from the analysis in Part B are given to the below (in green).



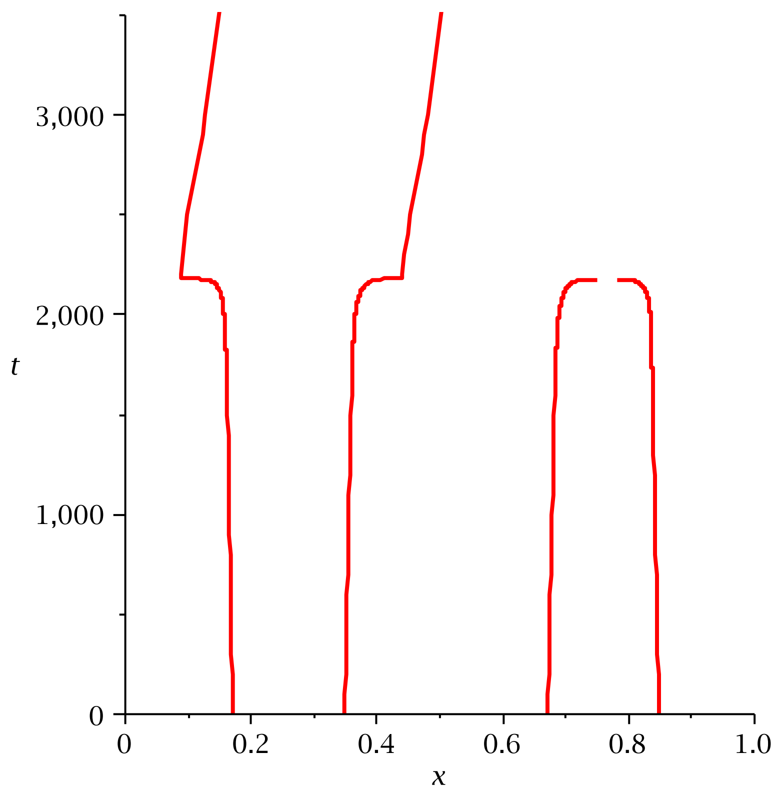


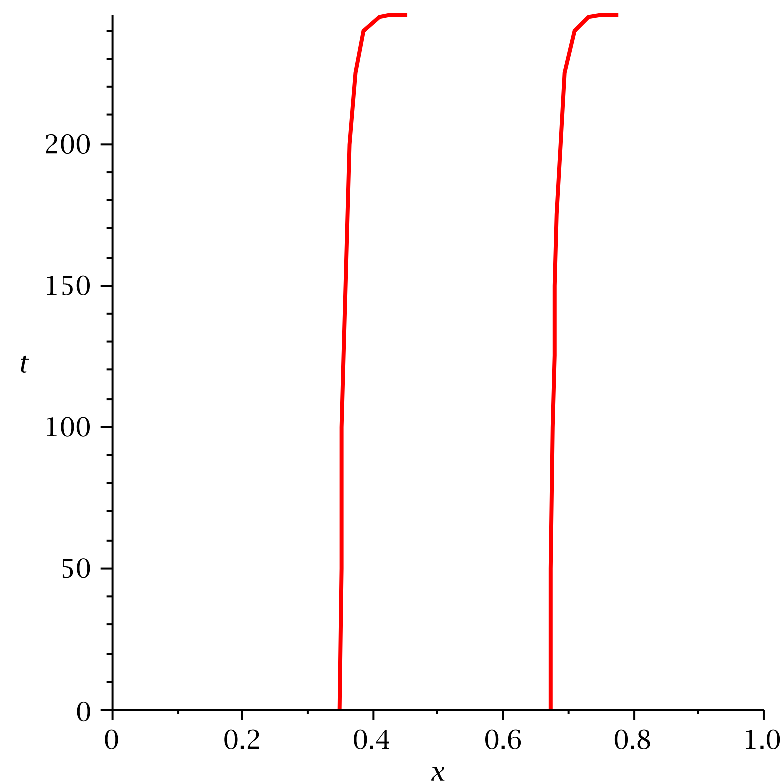
Some of the interesting dynamics that are seen in the two mesa case includes those seen in the following picture (x vs. t). Here, the two mesas that are seen initially (before $t = 250$) are unstable, but one mesa that is formed (after $t = 250$) is stable and moves towards the center of the interval. The parameters are $D = 2000, L = 1, \varepsilon = 0.1, \beta_0 = 1.3$.



Motion of the Interfaces for Two Mesas

Can the equation of motion determined in Part B describe the motion of the interfaces for two mesas? Unfortunately not. As we can see from the graphs below, the interfaces move on a completely different time scale, even though the same behaviour is exhibited. The same parameters are used in generating both graphs $D = 30$, $L = 0.25$,$\varepsilon = 0.02$, and $\beta_0 = 2$.





Future Work

* Determine the eigenvalues and eigenfunctions associated with the two mesa solution
* Describe all dynamics of the interfaces in the two mesa case
* Generalize these results for n mesas

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