On Tannaka Dualities

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What Tannaka duality is.

What I do.

What should be done.

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- Tannaka duality consists of reconstruction and representation.

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- Estimate the number of monoidal structures on the category of automata.

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What should be done.

- On fundamental theorem.
- On representation problem.

1. Tannaka Duality Theorem

Referrences on Tannaka dualtiy

Some referrences on Tannaka duality theorem and its generalizations.

- A. Joyal and R. Street, *An introduction to Tannaka duality and Quantum groups.*
- P. McCrudden, *Tannaka duality for Maschkean categories.*
- P. Deligne and J.S. Milne, *Tannakian Categories*.

Tannaka duality in $\mathbf{Vect}_k$

Taking representations

Given a coalgebra C in $\mathbf{Vect}_k$, one can construct the category $\mathbf{Rep}_f(C)$ of finite dimensional representations of C. Denote the forgetful functor by $F_C : \mathbf{Rep}_f(C) \to \mathbf{Vect}_k$.

Remark: representations of C = right C-comodules.

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Converse construction

Given $F : \mathbf{C} \to \mathbf{Vect}_k$, a functor s.t. $F(A)$ is finite dimensional, one can construct $C_F \in \mathbf{Vect}_k$, the coalgebra obtained by:

$$C_F = \int^{\tau \in \mathbf{C}} F(\tau)^* \otimes F(\tau) \tag{1}$$

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Constructively, this is constructed by taking an appropriate quatient space:

$$C_F = \left(\bigoplus_{\tau \in \mathbf{C}} F(\tau)^* \otimes F(\tau) \right) / \sim \tag{2}$$

Tannaka duality in $\mathbf{Vect}_k$

Fundamental Theorem of Coalgebras

A coalgebra in $\mathbf{Vect}_k$ is the union of its finite dimensional sub-coalgebras.

This is essentially because vectors in $C \otimes C$ is a *finite* sum of $c_1 \otimes c_2$.

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Theorem (Reconstruction theorem)

For an arbitrary coalgebra $C \in \mathbf{Vect}_k$, if $F : \mathbf{C} \to \mathbf{Vect}_k$ is the forgetful functor $F_C : \mathbf{Rep}_f(C) \to \mathbf{Vect}_k$, then we have an isomorphism:

$$C \xrightarrow{\simeq} C_{F_C} \tag{3}$$

Coend formula

A coalgebra can be reconstructed from its finite dimensional representations:

$$C = \int^{\tau \in \mathbf{Rep}_f(C)} F(\tau)^* \otimes F(\tau)$$

Tannaka duality in $\mathbf{Vect}_k$

Comparison functor

There is a canonical functor $\bar{F} : \mathbf{C} \longrightarrow \mathbf{Rep}_f(C_F)$ such that the following commutes:

$$\begin{array}{ccc} \mathbf{C} & \overset{\bar{F}}{\dashrightarrow} & \mathbf{Rep}_f(C_F) \\ & {\scriptstyle F}\searrow \quad \swarrow{\scriptstyle F_{C_F}} & \\ & \mathbf{Vect}_k & \end{array} \tag{4}$$

Remarkably, there is a characterization of fibre functors $F : \mathbf{C} \longrightarrow \mathbf{Vect}_k$ such that $\bar{F} : \mathbf{C} \longrightarrow \mathbf{Rep}_f(C_F)$ is an equivalence.

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Theorem (Representation theorem)

If $\mathbf{C}$ is k-linear abelian and F is exact and faithful, then $\bar{F}$ is an equivalence of categories (and vice versa).

Tannaka duality in $\mathbf{Vect}_k$

Main theme of Tannaka duality can be decomposed into the following two parts:

- Reconstruction problem:
  to reconstruct an algebraic structure from the category of its representations.
  - compact groups [Tannaka, '39], [Krein, '49]
  - locally compact groups [Tatsuuma, '67]
  - Hopf algebras [Ulbrich, '91]
  - quasi Hopf algebras [Majid, '92] etc.
- Representation problem:
  to characterize what category is equivalent to a category of representations of an algebraic structure.
  - pro-algebraic groups [Deligne and Milne, '81] : Tannakian category
  - compact groups [Doplicher and Roberts, '89]

The following universality of a coalgebra is important:

Universality of Coalgebra

$$C = \int^{\tau \in \mathbf{Rep}_f(C)} F(\tau)^* \otimes F(\tau)$$

because this universality shows several correspondences between structures on $\mathbf{Rep}_f(C)$ and those on C.

Tannaka duality in $\mathbf{Vect}_k$

Bialgebra structures induce monoidal structures.

Multiplication to monoidal structure

Given a bialgebra structure (μ, η) on a coalgebra $C \in \mathbf{Vect}_k$, one can construct a monoidal structure $(\otimes_\mu, I_\eta)$ on $\mathbf{Rep}_f(C)$, s.t. the forgetful functor $F_C : \mathbf{Rep}_f(C) \to \mathbf{Vect}_k$ is monoidal.

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Conversely, we have the inverse construction due to the universality of coalgebras.

Monoidal structure to multiplication

Given a functor $F : \mathbf{C} \to \mathbf{Vect}_k$ and a monoidal structure $(\otimes, I)$ on $\mathbf{C}$ s.t. F is monoidal, one can construct a bialgebra structure $(\mu_\otimes, \eta_I)$ on C_F.

Remark : We mean strong monoidal by "monoidal".

Remark : The non-strong case is also studied in, e.g., [Majid, '92].

Antipodes induce left dual objects.

Antipode to duals

If a bialgebra $B \in \mathbf{Vect}_k$ has its antipode $S : B \to B$, then the monoidal category $\mathbf{Rep}_f(B)$ has left dual objects.

Tannaka duality in $\mathbf{Vect}_k$

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The converse is also true.

Dual to antipode

Given a monoidal functor $F : \mathbf{C} \to \mathbf{Vect}_k$ s.t. $\mathbf{C}$ has left dual objects, then the bialgebra C_F is a Hopf algebra.

Especially..

The monoidal category $\mathbf{Rep}_f(B)$ has left dual objects if and only if B is a Hopf algebra.

Some generalizations of Tannaka duality theorem

There are known several directions to generalize Tannaka duality theorem and its analogues.

- *Tannakian categories* [P. Deligne and J. Milne, '82]
- *Tannaka duality for Maschkean categories* [P. McCrudden, '02]
- *Enriched Tannaka reconstruction* [B. Day, '96]

2. Discrete Analogue of Tannaka Duality

Tannaka duality in **Rel**

This study is originaly aimed at solving the following classification problem.

Original Problem

- How many monoidal structures can exist on the category $\mathbf{Aut}(\Sigma)$ of automata and simulations?
- Are there infinitely many monoidal structures?
- Can we give a good classification of them?

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The motivation comes from the following recent approach to concurrency theory based on categorical framework of state-based systems.

Motivation

"The microcosm principle and concurrency in coalgebra" [Jacobs et al, '08]

- Understand several existing constructions on state-based systems as categorical operations on particular category of universal coalgebra.

Tannaka duality in **Rel**: Reconstruction problem

Given a Hopf algebra $H \in \mathbf{Rel}$, we have following universality of H.

(Almost trivial) Universality of H

$$H = \int^{\tau \in \mathbf{Rep}(H)} F(\tau)^* \otimes F(\tau) \tag{5}$$

But this expression is not satisfactory because **Rel** is neither complete nor cocomplete. In fact:

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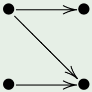
Lack of (co-) equalizers

$X = \{\bullet, \bullet\} \in \mathbf{Rel}$ and consider the following relation $f : X \to X$:



Then there is no equalizer for f and the identity $id_X : X \to X$.

Reconstruction problem

Tannaka duality theorem (Reconstruction of compact groups)

G: compact group, $\mathbf{Rep}_f(G, \mathbb{C})$: category of fin. dim. rep. of G.
$F : \mathbf{Rep}_f(G, \mathbb{C}) \to \mathbf{Vect}_k =$ the forgetful functor. Let $T(G) \subseteq \mathrm{End}(F)$ be a subset of natural trasformations $F \Rightarrow F$ satisfying:

$$\begin{aligned} U(\tau \otimes \rho) &= U(\tau) \otimes U(\rho) \\ U(I) &= id_I \\ \bar{U} &= U \end{aligned}$$

Then $T(G)$ forms a topological group and is canonically isomorphic to G.

Similar construction is known also for pro-algebraic groups [Deligne-Milne].

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Reconstruction via natural transformations

Can we reconstruct $H \in \mathbf{Rel}$ by using some class of natural transformations $F_H \Rightarrow F_H$ on the forgetful functor $F_H : \mathbf{Rep}(H) \to \mathbf{Rel}$?

Reconstruction problem

C: arbitrary monoidal category with left dual objects.
$F : \mathbf{C} \to \mathbf{Rel}$: a (strict) monoidal functor.

Poset structure on $\mathrm{End}(F)$

Given $U, V : F \Rightarrow F$, we denote by $U \leq V$ if for each $\tau \in \mathbf{C}$,

$$U(\tau) \subseteq V(\tau)$$

Remark : $\mathrm{End}(F) \ni U : F \Rightarrow F$ consists of $U(\tau) \subseteq F(\tau) \times F(\tau)$.

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Conjugate operator on $\mathrm{End}(F)$

Given $U \in \mathrm{End}(F)$, the conjugate $\bar{U} : F \Rightarrow F$ is defined: for each $\tau \in \mathbf{C}$, the component on τ is given by,

$$\bar{U}(\tau) = (U(\tau^*))^*$$

Remark : The internal $*$ is dual in **C**, and the external $*$ is dual in **Rel**.

Reconstruction problem

Especially, there is the minimal element $0 : F \Rightarrow F$ whose components are empty sets $0(\tau) = \emptyset \subseteq F(\tau) \times F(\tau)$.

Atoms in $\mathrm{End}(F)$

A natural transformation $U : F \Rightarrow F$ is called an atom if for every V, $V \leq U$ implies that V is equal to either 0 or U.

Denote by $H_F \subseteq \mathrm{End}(F)$ the set of all atoms in $\mathrm{End}(F)$.

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Some relations on H_F

$$
\begin{aligned}
H_F \times (H_F \times H_F) \supseteq \Delta_F &= \{(U,(V,W)) \mid U \leq W \circ V\} \\
(H_F \times H_F) \times H_F \supseteq \mu_F &= \{((U,V),W) \mid U \otimes V \leq W\} \\
H_F \times I \supseteq \epsilon_F &= \{(U,*) \mid U \leq id_F\} \\
I \times H_F \supseteq \eta_F &= \{(*,U) \mid \forall V,W.\ V \otimes U \leq W \Rightarrow V \leq W\} \\
H_F \times H_F \supseteq S_F &= \{(U,V) \mid U \leq \bar{V}\}
\end{aligned}
$$

Reconstruction problem

This structure gives a reconstruction of Hopf algebras in **Rel**.

Theorem (Reconstruction theorem)

If $F : \mathbf{C} \to \mathbf{Rel}$ is $F_H : \mathbf{Rep}(H) \to \mathbf{Rel}$ for some $H \in \mathbf{Rel}$, then there is a canonical isomorphism of Hopf algebras:

$$H \simeq H_{F_H}$$

We describe a sketch of the proof.

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We describe a sketch of the proof.

Notation

Let H be a Hopf algebra in **Rel** and $\tau = (X \to X \otimes H) \in \mathbf{Rep}(H)$.

$$x \xrightarrow{a} x' \quad \Leftrightarrow \quad (x, (x', a)) \in \tau$$

Remark : $\tau \subseteq X \times (X \times H)$.

Sketch of the proof.

Lemma. 1 (Comultiplication)

For every $a, b, c \in H$, we have:

$$(a,(b,c)) \in \Delta \quad \Leftrightarrow \quad \begin{cases} \forall \tau = (X \to X \otimes H) \in \mathbf{Rep}(H), \\ x \xrightarrow{a} x' \Rightarrow \exists x''.\ x \xrightarrow{b} x'' \xrightarrow{c} x' \end{cases}$$

Remark :$\Delta \subseteq H \times (H \times H)$.

Lemma. 2 (Multiplication)

For every $a, b, c \in H$, we have:

$$((a,b),c) \in \mu \Longleftrightarrow \begin{cases} \forall \tau = (X \to X \otimes H), \forall \rho = (Y \to Y \otimes H) \in \mathbf{Rep}(H), \\ x \xrightarrow{a} x' \ \text{ in } \tau \ \wedge \ \ y \xrightarrow{b} y' \ \text{ in } \rho \\ \Rightarrow x \otimes y \xrightarrow{c} x' \otimes y' \ \text{ in } \tau \otimes \rho \end{cases}$$

Remark : The underlying set of $\tau \otimes \rho$ is given by $X \times Y = X \otimes Y$. We denote $(x, y) \in X \otimes Y$ by $x \otimes y$.

Sketch of the proof

Lemma. 3 (Antipode)

For every $a, b \in H$, we have:

$$(a,b) \in S \Longleftrightarrow \begin{cases} \forall \tau = (X \to X \otimes H) \in \mathbf{Rep}(H), \\ x \xrightarrow{a} x' \ \text{ in } \tau \ \Rightarrow x' \xrightarrow{b} x \ \text{ in } \tau^* \end{cases}$$

Remark : The underlying set of τ^* is also $X(= X^*)$ for $\tau = (X \to X \otimes H)$.

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Lemma. 3 (Antipode)

For every $a, b \in H$, we have:

$$(a, b) \in S \iff \begin{cases} \forall \tau = (X \to X \otimes H) \in \mathbf{Rep}(H), \\ x \xrightarrow{a} x' \ \text{in } \tau \ \Rightarrow x' \xrightarrow{b} x \ \text{in } \tau^* \end{cases}$$

Remark : The underlying set of τ^* is also $X(= X^*)$ for $\tau = (X \to X \otimes H)$. We restate these lemmas in terms of natural transformations. To do so, we need the following notation.

Notation

For $a \in H$, a natural transformation $U_a : F \Rightarrow F$ is defined: for each $\tau = (X \to X \otimes H)$, the component $U_a(\tau) \subseteq X \times X$ is given by,

$$U_a(\tau) = \{(x, x') \mid x \xrightarrow{a} x' \ \text{in } \tau\}$$

Sketch of the proof

Proposition. 1 (Comultiplication)

For every $a, b, c \in H$, we have:

$$(a, (b, c)) \in \Delta \iff U_a \leq U_c \circ U_b$$

Sketch of the proof

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Proposition. 2 (Multiplication)

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$$((a, b), c) \in \mu \iff U_a \otimes U_b \leq U_c$$

Proposition. 3 (Antipode)

For every $a, b \in H$:

$$(a, b) \in S \iff U_a \leq \bar{U}_b$$

Sketch of the proof

In the case of compact group G.. [Joyal-Street]

A natural transformation $U : F \Rightarrow F$ on $F : \mathbf{Rep}_f(G, \mathbb{C}) \to \mathbf{Vect}_k$ is of the form $\pi(x)$ for some $x \in G$ if and only if U is self-conjugate and tensor-preserving.

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The notion of atoms characterizes U_a.

Proposition. 4

A natural transformation $U : F_H \Rightarrow F_H$ on $F_H : \mathbf{Rep}(H) \to \mathbf{Rel}$ is of the form U_a for some $a \in H$ if and only if U is an atom in $\mathrm{End}(F_H)$.

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Thus now we can describe the canonical isomorphism from H to H_{F_H}:

Canonical isomorphism

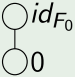
The canonical isomorphism is explicitly given by the following correspondence:

$$U_\bullet : H \ni a \mapsto U_a \in H_{F_H}$$

How to use this construction

Example (canonical embedding Sets → Rel)

Let F_0 : **Sets** → **Rel** be the canonical embedding, then the poset $\mathrm{End}(F_0)$ is isomorphic to the poset represented by the following Hasse diagram:



Thus H_{F_0} is a singleton $\{id_{F_0}\}$.

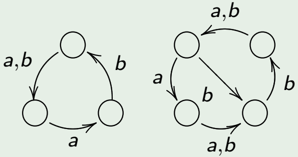
Some consequences for original problem

We do not forget the original problem.

Original problem

How many monoidal structures can exist on $\mathbf{Aut}(\Sigma)$? Are they finite or infinite? Can we give a good classification of them?

Rough description of $\mathbf{Aut}(\Sigma)$

Objects: 

... non-deterministic automata.

Arrows: (Backward-forward) simulations.

Some consequences for original problem

Typical monoidal structures on $\mathbf{Aut}(\Sigma)$

- CCS-like parallel composition of automata.
- CSP-like parallel composition of automata.
- Interleaving composition of automata.

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There is a functor $F : \mathbf{Aut}(\Sigma) \to \mathbf{Rel}$ that sends an automaton to its state-set, and a simulation to itself. These typical monoidal structures make $F : \mathbf{Aut}(\Sigma) \to \mathbf{Rel}$ strict monoidal.

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Restricted classification problem

Classify monoidal structures on $\mathbf{Aut}(\Sigma)$ such that $F : \mathbf{Aut}(\Sigma) \to \mathbf{Rel}$ is strict monoidal.

Remark : In what follows, "monoidal structure" means such monoidal structures.

Some consequences for original problem

Classification of monoidal structures

There is a bijective correspoindence between monoidal structures on $\mathbf{Aut}(\Sigma)$ and bialgebra structures on the coalgebra Σ^* consisting of finite words.

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The interleaving composition on $\mathbf{Aut}(\Sigma)$ is in correspondence with the shuffling operation on finite words under the above bijective correspondence.

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Example (Interleaving v.s. word shuffling)

The interleaving composition on $\mathbf{Aut}(\Sigma)$ is in correspondence with the shuffling operation on finite words under the above bijective correspondence.

Corollary: $\mathbf{Aut}(\Sigma)$ has only finitely many monoidal structures.

If the set Σ consists of n members, then the number $M(n)$ of monoidal structures on $\mathbf{Aut}(\Sigma)$ is finite: there is a rough estimation,

$$n! \leq M(n) \leq 2^{n^3+n}.$$

Some consequences for original problem

One can prove the following fact by combinatorial argument on finite words.

Lemma: Σ^* can not be a Hopf algebra in Rel.

The coalgebra Σ^* can not be a Hopf algebra with respect to any bialgebra structure on it.

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The coalgebra Σ^* can not be a Hopf algebra with respect to any bialgebra structure on it.

This fact is translated to a fact about $\mathbf{Aut}(\Sigma)$ via Tannaka dualtiy.

Corollary: $\mathbf{Aut}(\Sigma)$ cannot be autonomous.

More strongly: for any monoidal structure on $\mathbf{Aut}(\Sigma)$, there exists an automaton that does not have its left dual.

Some consequences for original problem

Automata are representations of finite words.

For $F : \mathbf{Aut}(\Sigma) \to \mathbf{Rel}$, we have an equivalence: $\mathbf{Aut}(\Sigma) \simeq \mathbf{Rep}(H_F)$:

- $H_F = \Sigma^*$: the set of finite words.
- $\Delta_F = \{(u, (v, w)) \mid u = v \cdot w\} \subseteq H_F \times (H_F \times H_F)$

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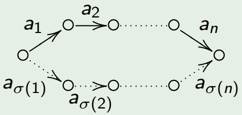
Example: Automata with permutable paths

$\mathbf{C} \subseteq \mathbf{Aut}(\Sigma)$: the full subcategory consisting of automata such that for each $\sigma \in \mathfrak{S}_n$,



For the restriction $F : \mathbf{C} \to \mathbf{Rel}$, we have an equivalence $\mathbf{C} \simeq \mathbf{Rep}(H_F)$.

- H_F: the set of multisets
- $\Delta_F = \{(p, (q, r)) \mid p = q + r\} \subseteq H_F \times (H_F \times H_F)$

4. Some Conjectures

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Observation

- **Rel** can be embedded into the category **SLat**
- $\otimes$ on **Rel** can be extended to $\otimes$ on **SLat**.
- **SLat** is complete and cocomplete.

Some Conjectures

Lesson from these observation

The place we should work in is not **Rel**, but **SLat** (or something like that).

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Correspondence

- The category $\mathbf{Vect}_k^f$ of finite dimensional spaces is repraced by **Rel**.
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Conjecture: Fundamental theorem in SLat

For a coalgebra $C \in \mathbf{SLat}$:

$$C = \int^{\tau \in \mathbf{Rep}_f(C)} F(\tau)^* \otimes F(\tau)$$

where $\mathbf{Rep}_f(C)$ consists of representations of C whose underlying set is in **Rel**, and $F : \mathbf{Rep}_f(C) \to \mathbf{SLat}$ denotes the forgetful functor.

Some Conjectures

Significant point of Tannaka duality

Starting from a category $\mathbf{C}$ which seemingly has nothing to do with coalgebras, one can prove an equivalence of $\mathbf{C}$ and the category $\mathbf{Rep}_f(C)$ of some coalgebra C.

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Conjecture (hope)

There is a category **Game** of some kind of games and a functor $F : \mathbf{Game} \to \mathbf{SLat}$ with $F(\tau)$ in **Rel**, such that $\mathbf{Game} \simeq \mathbf{Rep}_f(C_F)$.

Thank you!